On the controllability of a Cubic Semi-Linear Wave Equation

Carlos Barrón-Romero[1]

***Abstract*— A numerical method for a nonlinear control optimization of the open problem of the semi-linear wave equation stated by Jacques-Louis Lions (1928-2001) is depicted using the method of Glovinski-Lions-He. The goal of the numerical control depicted in this paper is to drive the state variable close to a given target function at finite time. Which is relevant because this equation without control decays or explodes. The results of the numerical experiments of a set of different target functions are satisfactory for the controllability and stability for an ad-hoc real control problem.**

I. INTRODUCTION

A cubic semi-linear state system:

$$y_{tt} - y_{xx} + y^3 = 0$$

had been studied for periodic solutions by Lidskii and Shulman [1]. Also, Donninger, R. and Zenginoğlu [2] studied an hyperboloidal initial value problem for the decaying of cubic wave equation:

$$(-\partial_t^2 + \triangle_x)v(t,x) + v(t,x)^3 = 0,\ x \in \mathbb{R}^3.$$

Enrique ZuaZua depicts the open problem of the exact controllability of the semi-linear wave equation stated by Jacques-Louis Lions [6], [9]:

$$\begin{cases} y_{tt} - y_{xx} + y^3 = \chi_\omega u(t,x) & (0,1)\times(0,T) \\ y = 0 & t \in (0,T) \\ y(0) = y_0 & x \in (0,1) \\ y_t(0) = y_1 & x \in (0,1) \end{cases}$$

where $T > 0$ and $\omega \subset (0,1)$, λ_ω is the characteristic function of the set ω, the state space is $(y(t,\cdot), y_t(t,\cdot))$ and u(t) is the control that acts on the system through ω. State variable y behaves in cubic form, it grows and shrinks fast, it well known that such kind of non-linear system is an open problem to study.

A weak controllability problem is

$$\text{find } u \in \mathcal{L}^2\left((0,T)\times(0,1)\right) \text{ such that y satisfies}$$

$y(0) = z_0$ and $y_t(T) = z_1$ in $(0,1)$, where (y_0, y_1) and $(z_0, z_1) \in H_0^1(0,1)\times\mathcal{L}^2(0,1)$ are given.

To my knowledge this is the first time that the Control Variational Methods of Glowinski-Lions-He [4], [7] are used to find numerical solutions of the previous control problem over a finite set of controls points distributed on (0,1). Other approach is given by Zhou, Xu, and Lei [3].

The appropriate control problem is stated in the next section. The section III depicts the construction of an ad-hoc version of the Conjugate Gradient Algorithm and an how to solve numerically a cubic polynomial for estimating the optimal numerical control for the discretization of state variable y^n evolving under the cubic semi-linear wave equation. A set of numerical experiments are presented in sectionIV. Finally, the last section presents the conclusions.

Professor Roland Glowinski thank you for your generosity and teachings.

[1]Basic Science Department, Engineering, and Basic Science Division, Autonomous Metropolitan University, Campus Azcapotzalco, Av. San Pablo No. 180, Col. Reynosa Tamaulipas, México, D.F. 02200 cbarron@azc.uam.mx

II. THE CONTROL PROBLEM

The formulation of the real control problem for the cubic semi-linear state equation is:

$$\begin{cases} \text{Find } v \in \boldsymbol{U} = \mathcal{L}^2([0,T],(0,1))^M \\ \mathrm{J}(v) \le \mathrm{J}(u), \forall u \in \boldsymbol{U} \end{cases} \qquad \text{(PCL)}$$

where M is the number of controls points in $(0,1)$ of the functional J:

$$\begin{aligned} \mathrm{J}(v) &= \frac{1}{2}\int_0^T ||v||^2 \mathrm{d}t + \frac{k_1}{2}\int_0^1 ||y(T) - z_0||^2 \mathrm{d}x \\ &+ \frac{k_2}{2}\int_0^1 ||\varphi_x||^2 \mathrm{d}x, \end{aligned}$$

for the control state system:

$$\begin{aligned} y_{tt} - y_{xx} + y^3 = \sum_{j=1}^{M} v_j \delta(x - a_j), \quad &\text{(EDy)} \\ y(0,t) = y(1,t) = 0, & \\ y(0) = y_0, & \\ y_t(x,0) = y_1(x), & \\ & \\ -\varphi_{xx} = y_t(T) - z_1, \quad &\text{(EDz)} \\ \varphi(0) = \varphi(1) = 0 & \end{aligned}$$

where $z_0 : [0,1] \to \mathbb{R}, z_1 : [0,1] \to \mathbb{R}, y_1 : [0,1] \to \mathbb{R}, a_j \in [0,1], j = 1,\ldots,M$.

The convergence and unique optimal solution for (PCL) under the techniques of Glowinski-Lions-He is a consequence of the formulation as a convex optimal problem on a appropriate complete Hilbert Spaces and the Lax-Milgram Theorem [4], [7].

III. BUILDING A CONJUGATE GRADIENT ALGORITHM

The steps to build an ad-hoc Conjugate Gradient (CG) Algorithm for (PCL) are:

1. Build a time discretization for (PCL).
2. Calculate $\delta\mathrm{J}^{\Delta t}$.

3. Build a fully discretization and find the partial differential subproblems to compute $\delta \mathrm{J}'^{\Delta t}$.

The discretization on time of J is

$$\begin{aligned} \mathrm{J}^{\Delta t}(v) &= \frac{\Delta t}{2}\sum_{n=0}^{N}||v^n||^2 + \frac{k_1}{2}\int_0^1 ||y^N - z_0||^2 \mathrm{d}x \\ &+ \frac{k_2}{2}\int_0^1 ||\varphi_x||^2 \mathrm{d}x \end{aligned}$$

where $N \in \mathbb{N}$, $\Delta t = \frac{T}{N}$.

Now, the discretization for (EDy) and (EDz) are:

$$\begin{aligned} \frac{y^{n+1} - 2y^n + y_{tt}^{n-1}}{(\Delta t)^2} - y_{xx}^n + (y^n)^3 &= \\ \sum_{j=1}^{M} v_j^n \delta(x - a_j), &\quad \text{(EDyn)} \\ y^n(0) &= 0 \\ y^n(1) &= 0, \\ y^0 &= y_0, \\ y_t^0 &= y_1. \\ -\varphi_{xx} = \frac{y^{N+1} - y^{N+1}}{2\Delta t} - z_1, &\quad \text{(EDzn)} \\ \varphi(0) &= 0 \\ \varphi(1) &= 0. \end{aligned}$$

The estimation of

$$\delta \mathrm{J}^{\Delta t}(u) = \left(\mathrm{J}'^{\Delta t}(u), w \right)_{U} = \Delta t \sum_{n=1}^{N} \mathrm{J}'^{\Delta t}(u)^n \cdot w^n$$

is by using the norm of $(\mathbb{R}^N)^M$.

Now, $\delta \mathrm{J}^{\Delta t}(u) = \Delta t \sum_{n=1}^{N} u^n \delta u^n + k_1(y^N - z_0)\delta y^N + k_2 \left(\frac{y^{N+1} - y^{N-1}}{2\Delta t} - z_1 \right) \frac{\delta y^{N+1} - \delta y^{N-1}}{2\Delta t}$.

The next step is for calculating the feedback of a variation of (EDyn) and (EDzn), multiplying by discrete variation function p^n and integrate them by part.

$$\begin{aligned} \frac{\delta y^{n+1} - 2\delta y^n + \delta y^{n-1}}{\Delta t^2} - \delta y_{xx}^n + \frac{(y^n)^2}{3}\delta y^n & \\ = \sum_{j=1}^{M} \delta(x - a_j)\delta v_j^n, &\quad (\delta\text{EDyn}) \\ \delta y^n(0) &= 0 \\ \delta y^n(1) &= 0, \\ \delta y^0 &= 0, \\ \delta y_t^0 &= 0. \\ -\delta\varphi_{xx} = \frac{\delta y^{N+1} - \delta y^{N+1}}{2\Delta t}, &\quad (\delta\text{EDzn}) \\ \delta\varphi(0) &= 0 \\ \delta\varphi(1) &= 0. \end{aligned}$$

$\Delta t \sum_{n=1}^{N} \int_0^1 \left(\frac{\delta y^{n+1} - 2\delta y^n + \delta y^{n-1}}{\Delta t^2} - \delta y_{xx}^n + \frac{(y^n)^2}{3}\delta y^n \right) p^n \, \mathrm{d}x = \Delta t \sum_{n=1}^{N} \int_0^1 \sum_{j=1}^{M} \delta(x - a_j)\delta v_j^n \, p^n \, \mathrm{d}x$.

But,

$$\Delta t \sum_{n=1}^{N} \int_0^1 \sum_{j=1}^{M} \delta(x - a_j)\delta v_j^n p \, \mathrm{d}x = \Delta t \sum_{n=1}^{N} \sum_{j=1}^{M} \delta v_j^n \, p_j \, \mathrm{d}x.$$

The explicit form is

$$\mathrm{J}'^{\Delta t}(v) = \{v_j^n + p_j^n\}_{n=1}^N.$$

$\Delta t \sum_{n=1}^{N} \int_0^1 \left(\frac{\delta y^{n+1} - 2\delta y^n + \delta y^{n-1}}{\Delta t^2} - \delta y_{xx}^n + \frac{(y^n)^2}{3}\delta y^n \right) p \, \mathrm{d}x = \Delta t \int_0^1 \left(\frac{\delta y^{N+1} - 2\delta y^N + \delta y^{N-1}}{\Delta t^2} - \delta y_{xx}^N + \frac{(y^N)^2}{3}\delta y^N \right) p \, \mathrm{d}x + \Delta t \int_0^1 \left(\frac{\delta y^{N} - 2\delta y^{N-1} + \delta y^{N-2}}{\Delta t^2} - \delta y_{xx}^{N-1} + \frac{(y^{N-1})^2}{3}\delta y^{N-1} \right) p \, \mathrm{d}x + \Delta t \int_0^1 \left(\frac{\delta y^{N-1} - 2\delta y^{N-2} + \delta y^{N-3}}{\Delta t^2} - \delta y_{xx}^{N-2} + \frac{(y^{N-2})^2}{3}\delta y^{N-2} \right) p \, \mathrm{d}x + \Delta t \sum_{n=1}^{N-2} \int_0^1 \left(\frac{\delta y^{n+1} - \delta 2 y^n + \delta y_{tt}^{n-1}}{\Delta t^2} - \delta y_{xx}^n + \frac{(y^n)^2}{3}\delta y^n \right) p \, \mathrm{d}x$.

From (δEDzn)

$-\delta\varphi_{xx} 2\,\Delta t + \delta y^{N-1} = \delta y^{N+1}$,

$-2\int_0^1 p^N \delta\varphi_{xx} \mathrm{d}x + \frac{1}{\Delta t}\int_0^1 \left(\frac{p^{N-1}}{\Delta t} - \frac{2\,p^N}{\Delta t} - \Delta t\, p_x^N x + \frac{\Delta t\, p^N\,(y^N)^2}{3} \right) \delta y^N \mathrm{d}x + \frac{1}{\Delta t}\int_0^1 \left(\frac{p^{N-2}}{\Delta t} - \frac{2\,p^{N-1}}{\Delta t} + \frac{p^N}{\Delta t} - \Delta t\, p_x^{N-1} x + \frac{\Delta t\, p^{N-1}\,(y^{N-1})^2}{3} \right) \delta y^N \mathrm{d}x + \frac{1}{\Delta t}\sum_{n=0}^{N-3} \int_0^1 \left(\frac{p^n - 2\,p^{n+1} + p^{n+2}}{\Delta t} - \Delta t\, p_{xx}^{n+1} + \Delta t \frac{(y^{n+1})^2}{3}\, p^{n+1} \right) \delta y^n \mathrm{d}x + \int_0^1 \left(\frac{-2\,p^0 + p^1}{\Delta t} - \Delta t\, p_{xx}^0 + \Delta t \frac{(y^0)^2}{3}\, p^0 \right) \delta y^0 \mathrm{d}x$,

$$-2\int_0^1 p^N \delta\varphi_{xx} \mathrm{d}x = 2\int_0^1 p_x^N \delta\varphi_x \mathrm{d}x.$$

Finally, the system of equations (EDp) for p^n are

$$2 p_x^N = k_2\, \varphi$$

$\frac{p^{N-1}}{\Delta t} - \frac{2\,p^N}{\Delta t} - \Delta t\, p_x^N x + \frac{\Delta t\, p^N\,(y^N)^2}{3} = k_1\,(y^N - z_0)$

$\frac{p^n - 2\,p^{n+1} + p^{n+2}}{\Delta t} - \Delta t\, p_{xx}^{n+1} + \Delta t \frac{(y^{n+1})^2}{3}\, p^{n+1} = 0,$

$n = N-2, \ldots, 1$

$\frac{-2\,p^0 + p^1}{\Delta t} - \Delta t\, p_{xx}^0 + \Delta t \frac{(y^0)^2}{3}\, p^0 = 0.$

A. *CG Algorithm*

A version of the CG algorithm for (PCL) has the following steps:

1) Let u_q^0.
2) Solve (Ed1n) and get y_q^0.
3) Solve (Ed2n) and get z_q.
4) With y_q and z_q solve (Edp) and get p_q.
5) Update $gq_j = uq_j + pq_j$.
6) Set $gq2 = ||gq||^2$, $y_{qb} = y_q$, $w_q = g_q$. if $\frac{||gq||^2}{\max(1, ||u_q||)} < \epsilon$ take u_0 like solution, otherwise, we have u_{q+1}, w_{q+1}, g_{q+1}.
7) With wq and y_{qb} solve (δEdpn), and get $\bar{y}_q$.
8) Now using $\bar{y}$ solve (δEDzn') and get $\bar{z}_q$.
9) Solve (δEdp') with y_{qb}, $\bar{y}_q$, and $\bar{z}_q$ and compute p_q.
10) Update $\bar{g}_{qj} = w_{qj} + p_{qj}$
11) $\bar{g}w_q = ||\bar{g}_q||\, ||w_q||$
12) $\rho_q = \frac{||g_q||^2}{\bar{g}_q w_q}$
13) $u_{q+1} = u_q - \rho_q\, w_q$
14) $g_{q+1} = g_q - \rho_q\, \bar{g}_q$ if $\frac{||g_{q+1}||^2}{||g_0||^2} < \epsilon$ take u_{q+1} as the solution, otherwise continue.

15) $\gamma_q = \frac{||g_{q+1}||^2}{||g_q||^2}$, $w_q = g_{q+1} + \gamma_q$, w_q. $q = q+1$, go to step 7.

This is a classical version of Conjugate Gradient Algorithm for solving the problem by residual (v. [4] p. 164). Looking carefully the steps 6 to 15 are for solving (δEDpn), (δEDzn') and (δEdp'). The details to solve the finite differences equations are depicted in the next section. A reference for finite differences techniques is [8].

B. Algorithm for (EDyn)

The main idea for solving the explicit formulation is by using the information of the two consecutive steps. It is possible because the initial condition for (EDy) are for time $n = 0$, and also for $n = 1$. The first is follow from $y(0) = y_0$ and the second is from $y_t(0) = y_1$. The first condition gives,

$$y_i^0 = y_0(x_i), \; i = 0, \ldots, N$$

For the second condition, the following central discrete approximation is considered:

$$y_t(0) \approx \frac{y_i^1 - y_i^{-1}}{2\,\Delta t}$$

and for $n = 1$,

$$\frac{2\,y_i^1 - 2\,y_i^0 - \Delta t y_i^1}{\Delta t^2} - \frac{y_{i+1}^0 - 2\,y_i^0 + y_{i-1}^0}{(\Delta x)^2} + \frac{1}{32}(2y_i^1 + 2y_i^0 - 2\,\Delta t y_i^1)((y_i^1)^2 + 2\,y_{0i}(2\,y_i^1 + y_i^0 - 2\,\Delta t y_{1i}) + (y_i^1 2\,\Delta t y_{1i})^2) = \sum_{j=1}^{M} \delta(x - a_j)\delta v_j^1$$

The formula for y^{n+1} is a polynomial of degree 3.

$$\frac{\Phi(\zeta_1) - \Phi(\zeta_2)}{\zeta_1 - \zeta_2} = \phi(\zeta_1) \;\; \text{if} \;\; ||\zeta_1 - \zeta_2|| \text{ is small}$$

where $\Phi' = \phi$.

In fact, for solving y_i^{n+1} for $n = 1, \ldots, N$

$y_i^0 = y_0(x_i), i = 0, \ldots, N$

$\frac{(y_i^{n+1})^3}{8} + \frac{y_i^n - \Delta t y_i^n}{8}(y_i^{n+1})^2 + (\frac{(y_i^n - \Delta t y_i^n)^2}{8} + \frac{(\Delta t y_i^n)^2}{8} + \frac{2}{(\Delta t)^2})y_i^{n+1}$

$-\frac{2\,\Delta t\, y_i^{n+1} + 2\,y_i^n}{(\Delta t)^2} - \frac{y_{i+1}^n - 2\,y_i^n + y_{i-1}^n}{(\Delta x)^2} + \sum_{j=1}^{M} \delta(x - a_j)\delta v_j^n = 0$.

Regrouping terms, the following cubic equation is obtained

$$a_n\,(y_i^{n+1})^3 + b_n\,(y_i^{n+1})^2 + c_n\,y_i^{n+1} + d_n = 0 \qquad (1)$$

where

$$\begin{aligned} a_n &= \frac{1}{8}, \\ b_n &= \frac{y_i^n - \Delta t y_i^n}{8}, \\ c_n &= (\frac{(y_i^n - \Delta t y_i^n)^2}{8} + \frac{(\Delta t y_i^n)^2}{8} + \frac{2}{(\Delta t)^2}), \\ d_n &= \frac{2\,\Delta t\, y_i^{n+1} + 2\,y_i^n}{(\Delta t)^2} - \frac{y_{i+1}^n - 2\,y_i^n + y_{i-1}^n}{(\Delta x)^2} \\ &+ \sum_{j=1}^{M} \delta(x - a_j)\delta v_j^n. \end{aligned}$$

For each i, y_i^{n+1} is a root of the equation (1). Such roots are estimated by the Newtow-Raphson Method.

IV. NUMERICAL EXPERIMENTS

The advantage of using the Glowinski-Lions-He's Methodology is that the Conjugate Gradient algorithm codification is straightforward from the description on section III-A.

The tolerance of the Newton-Raphson Method is 5×10^{-6}, the tolerance of CG algorithm is 5×10^{-6}, the number of points for the discretization of the space interval $(0, 1)$ is 66, the number of time steps is $N = 231$, and for all the numerical experiments $T = 1$; for J, $k_1 = 2 \times 10^6$ and $k_2 = 20$ and a_j denotes the position of the control points.

For each test problem, different controls points were selected in order to reach the target function $z_0(x) \neq 0$ with minimum error from the given initial state $y_0 = 0$ and $y_1(x) = 0$ at $t = 0$. The error is given by err$(y(T))$ = $||y(T) - z_0||$ and err$(y_t(T)) = ||y_t(T) - z_1||$. The last one means $(y_t) \approx 0$ at T.

The graphics of the numerical experiments depicts the controls points functions evolving over time, the overlapping between state function (y) and its corresponding target function (z_0), and the overlapping between time derivative of the state function (y_t) and the function $(z_1 = 0)$ at T.

Target function 1: $z_0(x) = -(x - 1)x^6$.
$M = 1$: err$(y(T)) = 2.32 \times 10^{-2}$, err$(y_t) = 5.62 \times 10^{-2}$, 6 steps. See fig. 1, $a_j \in \{0.85\}$. $M = 2$: err$(y(T)) = 7.41 \times 10^{-2}$, err$(y_t) = 6.02 \times 10^{-6}$, 32 steps. See fig. 2, $a_j \in \{0.37, 0.63\}$.

Target function 2: $z_0\,(x) = \begin{cases} x & x \in \left(0, \frac{1}{2}\right), \\ 1 - x & x \in \left(\frac{1}{2}, 1\right). \end{cases}$
$M = 1$: err$(y_t) = 7.53 \times 10^{-3}$, 2 steps. See fig. 3, $a_j \in \{0.5\}$, err$(y(T)) = 8.14 \times 10^{-2}$.

Target function 3: $z_0(x) = \sin(\pi x)$.
$M = 1$: err$(y(T)) = 1.58 \times 10^{-1}$, err$(y_t) = 8.75 \times 10^{-2}$, 2 steps. See fig. 5, $a_j \in \{0.5\}$. Fig. 4 depicts the evolution over the space and time of the state function y toward the target function.

Target function 4: $z_0(x) = 2\sin(4\pi x)^5$.
$M = 2$: err$(y(T)) = 7.49 \times 10^{-1}$, err$(y_t) = 4.3 \times 10^{-2}$, 2 steps. See fig. 7, $a_j \in \{0.37, 0.63\}$.

Fig. 6 depicts the evolution over the space and time of the state function y toward the target function.

Target function 5: $z_0(x) = \begin{cases} 0 & x \in (0, \frac{1}{2}) \cup (\frac{3}{4}, 1), \\ 1.5 & x \in (\frac{1}{2}, \frac{3}{4}). \end{cases}$

$M = 3$: $\text{err}(y(T)) = 1.99 \times 10^0$, $\text{err}(y_t) = 2.38 \times 10^{-1}$, 6 steps. See fig. 8, $a_j \in \{0.134, 0.5, 0.866\}$.

$M = 5$: $\text{err}(y(T)) = 1.92 \times 10^0$, $\text{err}(y_t) = 2.32 \times 10^{-1}$, 5 steps. See fig. 10, $a_j \in \{0.134, 0.2929, 0.5, 0.7071, 0.866\}$.

Target function 6: $z_0(x) = -(x-1)x^3$.

$M = 1$: $\text{err}(y(T)) = 1.88 \times 10^{-2}$, $\text{err}(y_t) = 9.19 \times 10^{-3}$, 2 steps. See fig. 10, $a_j \in \{0.85\}$.

$M = 5$: $\text{err}(y(T)) = 2.21 \times 10^{-2}$, $\text{err}(y_t) = 1.15 \times 10^{-4}$, 2 steps. See fig. 10, $a_j \in \{0.134, 0.2929, 0.5, 0.7071, 0.866\}$.

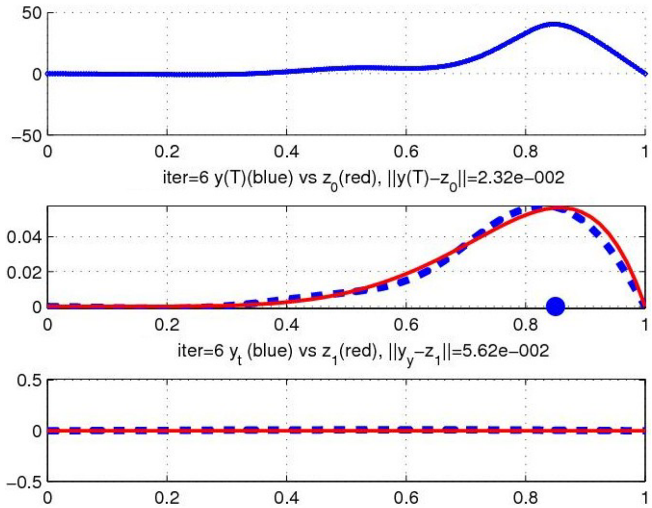

Fig. 1. $z_0(x) = -(x-1)x^6$, $M = 1$

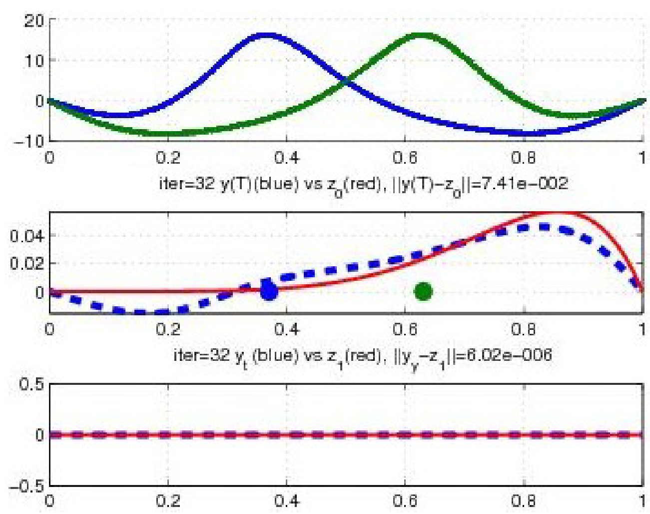

Fig. 2. $z_0(x) = -(x-1)x^6$, $M = 2$

V. CONCLUSIONS

Without controls, it is not possible to reach any of the target functions. The results depicted are the best considering the control's position for a low error between the target function and the state function at the final time T. The

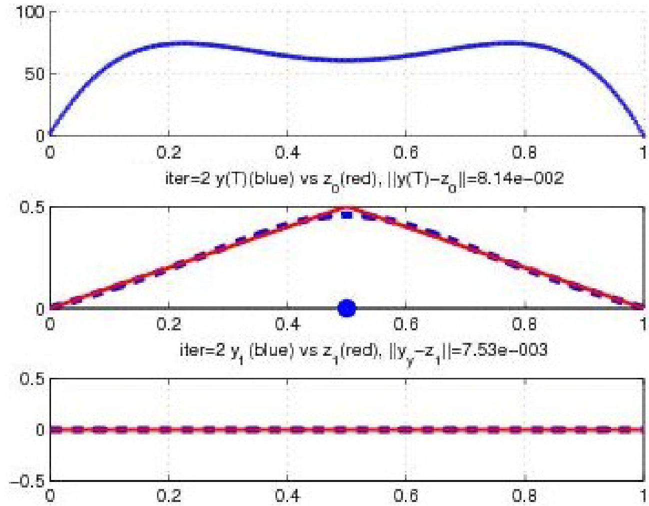

Fig. 3. Z_0 is a triangular pulse, $M = 1$

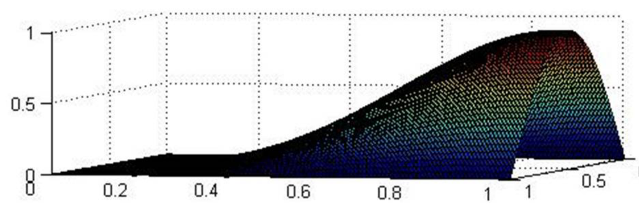

Fig. 4. Evolution of $y(x,t)$ toward $z_0(x) = \sin(\pi x)$, $M = 1$

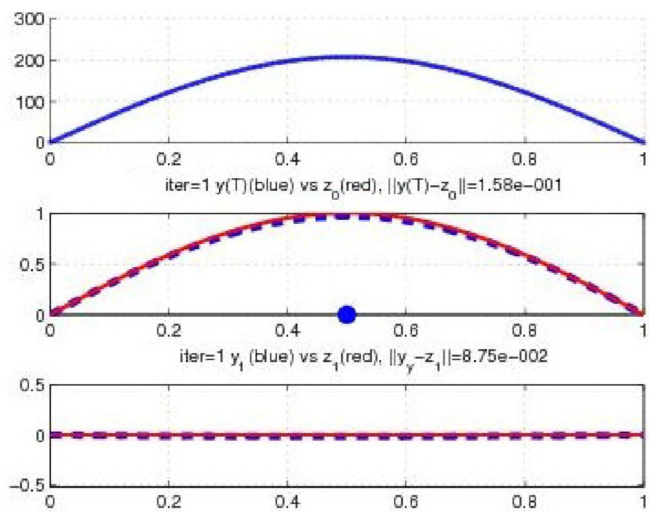

Fig. 5. $z_0(x) = \sin(\pi x)$, $M = 1$

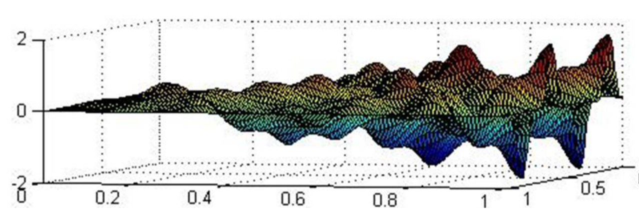

Fig. 6. Evolution of $y(x,t)$ toward $z_0(x) = 2\sin(4\pi x)^5$, $M = 2$

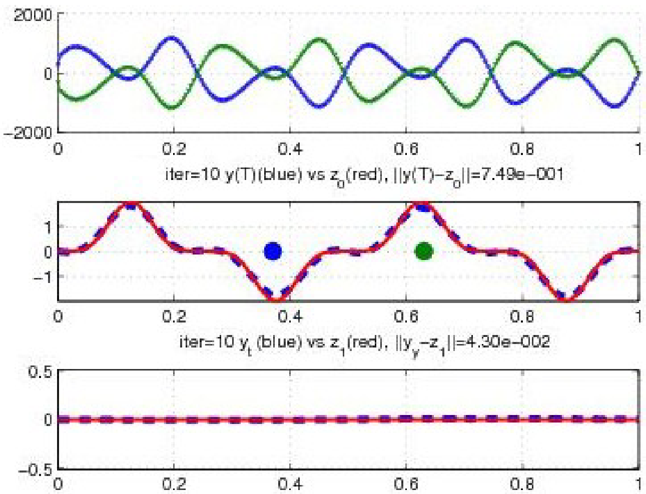

Fig. 7. $z_0(x) = 2\sin(4\pi x)^5$, $M = 2$

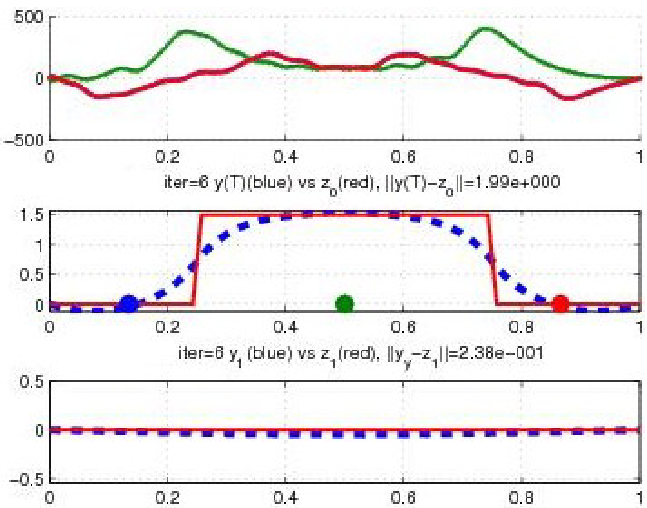

Fig. 8. $z_0(x)$ is a square pulse, $M = 3$

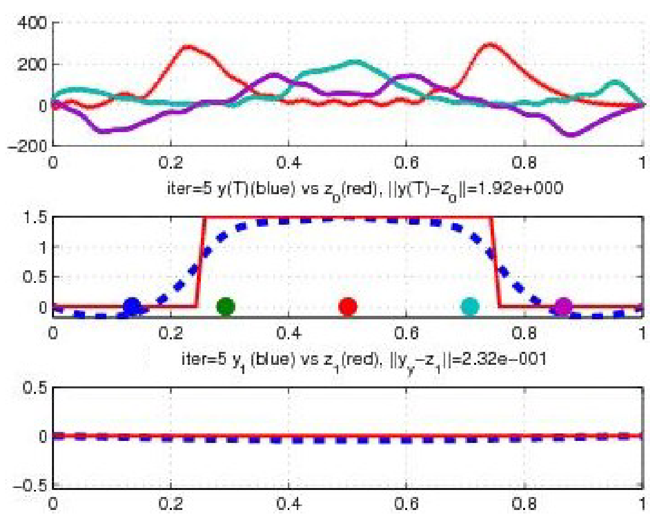

Fig. 9. z_0 is a square pulse, $M = 5$

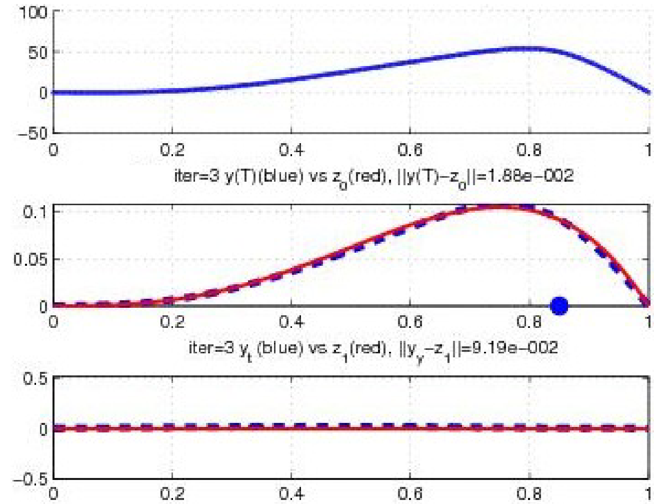

Fig. 10. $z_0(x) = -(x-1)x^3$, $M = 1$

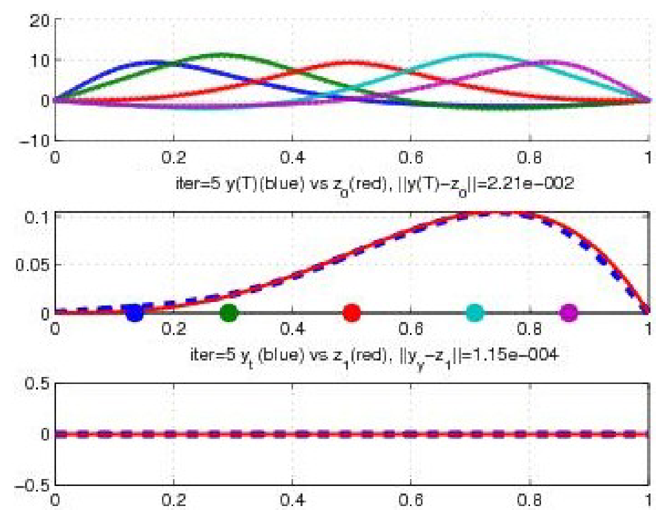

Fig. 11. $z_0(x) = -(x-1)x^3$, $M = 5$

summary of the the numerical best results depicts that in many smooth target functions, one controls is enough for reaching them. See figures 1, 3, 5, and 10. Other cases require more controls points, see figures 7, 8, and 9. Figures 1 and 2 depict that one control in an appropriate position reach the target function better than two controls. Figures 10 and 11 depict that there are not improvement for reaching the target function by using more controls points. In these two cases, the number of steps grow with the controls points but without a significant improvement on the final result. In the future there will be a more extensive study.

REFERENCES

[1] B. V. Lidskii, E. I. Shulman, Periodic solutions of the equation $u_{tt} - u_{xx} + u^3 = 0$ *Funktsional. Anal. i Prilozhen* 22-4, 88-89 (1988)

[2] Donninger, R. and Zenginoğlu, A. *Nondispersive decay for the cubic wave equation.* Analysis & PDE, 7, 461-495 (2014)

[3] ZHOU, Yi and XU, Wei and LEI, Zhen Global Exact Boundary Controllability for Cubic Semi-linear Wave Equations and Klein-Gordon Equations *Chinese Annals of Mathematics* 31B(1), 35-58 (2010)

[4] Glowinski, R. and Lions, J. L.: Exact and approximatte controllability for distributed parameter systems. *Acta Numerical*, 159–333 (1996)

[5] Dean, E. J., Glowinski, R., Kuo, Y., and Nasser, M.: On the discretization of some second order in time differential equations applications to nonlinear wave problems. The Proceedings Of The NASA-UCLA Workshop On Computational Thechniques In Identification And Control Of Flexible Flight Structures, (1989)

[6] Zuazua, E. *Jacques-Louis Lions: Hasta siempre*. Gaceta de la RSME, 5-2, 323-329 (2002)

[7] Glowinski, R., Lions, J. L., and He, J.: *Exact and Approximate Controllability for Distributed Parameter Systems*. Encyclopedia of Mathematics. Cambrigde University Press, Cambrigde (2008).

[8] Morton, K. W. and Mayers, D. F.: *Numerical Solution of Partial Differential Equations*. Cambrigde University Press, (1994)

[9] Zhang, X. and Zuazua, E.: *Exact controllability of the semilinear wave equation*. In: Blondel, V.D., Megretski, A. (eds.): Unsolved Problems in Mathematical Systems and Control Theory, pp. 173–178. Princeton Unversity Press, New Jersey (2004)