A Conjugated Gradient Algorithm to solve a Control Problem with Non Linear Partial Differential System

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1 Introduction

Here we are interesting to solve the following control problem:

$$\text{PCL} \quad \begin{cases} v^0 \in U = L^2((0,T), (\mathrm{R}^N)^M) \\ \mathrm{J}(v) \leq \mathrm{J}(u), \forall u \in U \end{cases}$$

where, [1]

$$\mathrm{J}(v) = \frac{1}{2}\int_0^T ||v||^2 \mathrm{d}t + \frac{k_1}{2}\int_0^L ||y(T) - z_0||^2 \mathrm{d}x + \frac{k_2}{2}\int_0^L ||\varphi_x||^2 \mathrm{d}x,$$

$$y_{tt} - y_{xx} + y^3 = f + \sum_{j=1}^{M} v_j \delta(x - a_j), \qquad \text{(EDy)}$$

$$\begin{aligned} y(0,t) &= y(L,t) = 0, \\ y(0) &= y_0 \\ y_t(x,0) &= y_1(x) \end{aligned}$$

$$-\varphi_{xx} = y_t(T) - z_1, \qquad \text{(EDz)}$$

$$\varphi(0) = \varphi(L) = 0,$$

We following the techniques of Glowinski and Lions [1] to solve optimal control problems with distributed parameters. We are interesting in the properties, like numerical stability, converge, and, of course, to find one optimal control. But, this is a preliminary report. In advance for future research, here we only show how build a Conjugated Gradient (CG) algorithm and show some numerical experiments and results.
In this report, we give the algoritmh on section 2, the numerical experiment and results on section 3, our conmments. By other hand, if you are interesting in our MatLab's source code, please send a e-mail to cbarron@math.uh.edu. We will glad to ear about you and we will send a copy.

2 Building a GC algoritmh

The steps to build a Gradient Conjugated algoritmh are:

[1] In all report, we use the convention $f(t) = g$ means $h(x,t) = g(x)$.

Step 1) Build the Time Discretization problem for PCL.

Step 2) Calculate δ $\mathrm{J}^{\Delta t}$.

Step 3) Build the fully Discretization and find the partial diferential subproblems to compute δ $\mathrm{J\prime}^{\Delta t}$.

The discretization on time of J is

$$\mathrm{J}^{\Delta t}(v) = \frac{\Delta t}{2}\sum_{n=0}^{N} ||v^n||^2 + \frac{k_1}{2}\int_0^L ||y^N - z_0||^2 \mathrm{d}x + \frac{k_2}{2}\int_0^L ||\varphi_x||^2 \mathrm{d}x,$$

where $N \in$ N, $\Delta t = \frac{T}{N}$.
Now, the discretization for EDy and EDz are:

$$\frac{y^{n+1} - 2y^n + y_{tt}^{n-1}}{(\Delta t)^2} - y_{xx}^n + (y^n)^3 = f^n + \sum_{j=1}^{M} v_j^n \delta(x - a_j), \qquad \text{(EDyn)}$$

$$y^n(0) = 0$$
$$y^n(L) = 0,$$
$$y^0 = y_0,$$
$$y_t^0 = y_1.$$

$$-\varphi_{xx} = \frac{y^{N+1} - y^{N+1}}{2\Delta t} - z_1, \qquad \text{(EDzn)}$$

$$\varphi(0) = 0$$
$$\varphi(L) = 0.$$

To calculate

$$\delta\ \mathrm{J}^{\Delta t}(u) = \Big(\ \mathrm{J\prime}^{\Delta t}(u), w\Big)_U = \Delta t \sum_{n=1}^{N}\ \mathrm{J\prime}^{\Delta t}(u)^n \cdot w^n$$

we use the standar norm of ($\mathrm{R}^N)^M$.
Now,

$$\delta\ \mathrm{J}^{\Delta t}(u) = \Delta t \sum_{n=1}^{N} u^n \delta u^n + k_1(y^N - z_0)\delta y^N + k_2 \frac{y^{N+1} - y^{N-1}}{2\Delta t} - z_1)\frac{\delta y^{N+1} - \delta y^{N-1}}{2\Delta t}$$

.
With this, we continue the second step, to calculate a variation of (EDyn) and (EDzn), multiply by discrete variation function p^n and integrate by part.

$$\frac{\delta y^{n+1} - 2\delta y^n + \delta y^{n-1}}{\Delta t^2} - \delta y^n_{xx} + \frac{(y^n)^2}{3}\delta y^n = \sum_{j=1}^{M} \delta(x - a_j)\delta v^n_j, \qquad (\delta \text{ EDyn})$$

$$\delta y^n(0) = 0$$

$$\delta y^n(L) = 0,$$

$$\delta y^0 = 0,$$

$$\delta y^0_t = 0.$$

$$-\delta\varphi_{xx} = \frac{\delta y^{N+1} - \delta y^{N+1}}{2\Delta t}, \qquad (\delta \text{ EDzn})$$

$$\delta\varphi(0) = 0$$

$$\delta\varphi(L) = 0.$$

$$\Delta t \sum_{n=1}^{N} \int_0^L (\frac{\delta y^{n+1} - 2\delta y^n + \delta y^{n-1}}{\Delta t^2} - \delta y^n_{xx} + \frac{(y^n)^2}{3}\delta y^n) p^n \,\mathrm{d}x = \Delta t \sum_{n=1}^{N} \int_0^L \sum_{j=1}^{M} \delta(x - a_j)\delta v^n_j \, p^n \,\mathrm{d}x.$$

But,

$$\Delta t \sum_{n=1}^{N} \int_0^L \sum_{j=1}^{M} \delta(x - a_j)\delta v^n_j p \,\mathrm{d}x = \Delta t \sum_{n=1}^{N} \sum_{j=1}^{M} \delta v^n_j \, p_j \,\mathrm{d}x.$$

We get a explicit form,

$$\mathrm{J}\prime^{\Delta t}(v) = \{v^n_j + p^n_j\}_{n=1}^{N}.$$

$$\Delta t \sum_{n=1}^{N} \int_0^L (\frac{\delta y^{n+1} - 2\delta y^n + \delta y^{n-1}}{\Delta t^2} - \delta y^n_{xx} + \frac{(y^n)^2}{3}\delta y^n) p \,\mathrm{d}x =$$

$$\Delta t \int_0^L (\frac{\delta y^{N+1} - 2\delta y^N + \delta y^{N-1}}{\Delta t^2} - \delta y^N_{xx} + \frac{(y^N)^2}{3}\delta y^N) p \,\mathrm{d}x +$$

$$\Delta t \int_0^L (\frac{\delta y^{N} - 2\delta y^{N-1} + \delta y^{N-2}}{\Delta t^2} - \delta y^{N-1}_{xx} + \frac{(y^{N-1})^2}{3}\delta y^{N-1}) p \,\mathrm{d}x +$$

$$\Delta t \int_0^L (\frac{\delta y^{N-1} - 2\delta y^{N-2} + \delta y^{N-3}}{\Delta t^2} - \delta y^{N-2}_{xx} + \frac{(y^{N-2})^2}{3}\delta y^{N-2}) p \,\mathrm{d}x +$$

$$\Delta t \sum_{n=1}^{N-2} \int_0^L (\frac{\delta y^{n+1} - \delta 2y^n + \delta y^{n-1}_{tt}}{\Delta t^2} - \delta y^n_{xx} + \frac{(y^n)^2}{3}\delta y^n) p \,\mathrm{d}x$$

From (δ EDzn),

$$-\delta\varphi_{xx} 2\,\Delta t + \delta y^{N-1} = \delta y^{N+1}$$

We get,

$$-2\int_0^L p^N \delta\varphi_{xx}\mathrm{d}x + \frac{1}{\Delta t}\int_0^L (\frac{p^{N-1}}{\Delta t} - \frac{2\,p^N}{\Delta t} - \Delta t\, p_x^N x + \frac{\Delta t\, p^N\,(y^N)^2}{3})\delta y^N \mathrm{d}x +$$
$$\frac{1}{\Delta t}\int_0^L (\frac{p^{N-2}}{\Delta t} - \frac{2\,p^{N-1}}{\Delta t} + \frac{p^N}{\Delta t} - \Delta t\, p_x^{N-1} x + \frac{\Delta t\, p^{N-1}\,(y^{N-1})^2}{3})\delta y^N \mathrm{d}x +$$
$$\frac{1}{\Delta t}\sum_{n=0}^{N-3}\int_0^L (\frac{p^n - 2\,p^{n+1} + p^{n+2}}{\Delta t} - \Delta t\, p_{xx}^{n+1} + \Delta t\frac{(y^{n+1})^2}{3}\, p^{n+1})\delta y^n \mathrm{d}x +$$
$$\int_0^L (\frac{-2\,p^0 + p^1}{\Delta t} - \Delta t\, p_{xx}^0 + \Delta t\frac{(y^0)^2}{3}\, p^0)\delta y^0 \mathrm{d}x$$

But,

$$-2\int_0^L p^N \delta\varphi_{xx}\mathrm{d}x = 2\int_0^L p_x^N \delta\varphi_x \mathrm{d}x$$

Finally, we get the sistem of equations (EDp) for p^n,

$$2p_x^N = k_2\,\varphi$$

$$\frac{p^{N-1}}{\Delta t} - \frac{2\,p^N}{\Delta t} - \Delta t\, p_x^N x + \frac{\Delta t\, p^N\,(y^N)^2}{3} = k_1\,(y^N - z_0)$$

$$\frac{p^n - 2\,p^{n+1} + p^{n+2}}{\Delta t} - \Delta t\, p_{xx}^{n+1} + \Delta t\frac{(y^{n+1})^2}{3}\, p^{n+1} = 0,\ n = N-2, \ldots, 1$$

$$\frac{-2\,p^0 + p^1}{\Delta t} - \Delta t\, p_{xx}^0 + \Delta t\frac{(y^0)^2}{3}\, p^0 = 0.$$

2.1 GC algorithm

The GC algorithm for (PLC) is

1. Let u_q^0.
2. Solve (Ed1n) and get y_q^0.
3. Solve (Ed2n) and get z_q.
4. With y_q and z_q solve (Edp) and get p_q.
5. Update $gq_j = uq_j + pq_j$.
6. Set $gq2 = ||gq||^2$, $y_{qb} = y_q$, $w_q = g_q$. if $\frac{||gq||^2}{\max(1,||u_q||)} < \epsilon$ take u_0 like solution, otherwise, we have u_{q+1}, w_{q+1}, g_{q+1}.

7. With wq and y_{qb} solve (δ Edpn), and get $\bar{y}_q$.

8. Now using $\bar{y}$ solve (δ EDzn') and get $\bar{z}_q$.

9. Solve (δ Edp') with y_{qb}, $\bar{y}_q$, and $\bar{z}_q$ and compute p_q.

10. Updte $\bar{g}_{qj} = w_{qj} + p_{qj}$

11. $\bar{g}w_q = ||\bar{g}_q||\,||w_q||$

12. $\rho_q = \frac{||g_q||^2}{\bar{g}_q w_q}$

13. $u_{q+1} = u_q - \rho_q\, w_q$

14. $g_{q+1} = g_q - \rho_q\, \bar{g}_q$ if $\frac{||g_{q+1}||^2}{||g_0||^2} < \epsilon$ take u_{q+1} like solution, otherwise continue.

15. $\gamma_q = \frac{||g_{q+1}||^2}{||g_q||^2}$, $w_q = g_{q+1} + \gamma_q$, w_q. $q = q+1$, repeat from step 7.

This is a clasical GC algorithm[2]. However it has slight different details to solve the partial diferential equations that we show in the next sections.

2.2 Algorithm for (EDyn)

The main idea is solve a explicit formulation to compute the solution in next step time using the information of the two previus step. It is possible becouse, the initial condition for (EDy) are for time $n = 0$, and also for $n = 1$. The first is follow from $y(0) = y_0$ and the second is from $y_t(0) = y_1$. The first give,

$$y_i^0 = y_0(x_i)\, i = 0, \dots, N$$

For the second we use the following central discrete aproximation:

$$y_t(0) \approx \frac{y_i^1 - y_i^{-1}}{2\,\Delta t}$$

and for $n = 1$ we have

$$\begin{aligned} \frac{2\,y_i^1 - 2\,y_{0i} - \Delta t y_{1i}}{\Delta t^2} - \frac{y_{i+1}^n - 2\,y_i^n + y_{i-1}^n}{(\Delta x)^2} + \\ \frac{1}{32}(2y_i^1 + 2y_i^0 - 2\,\Delta t y_{1i})((y_i^1)^2 + \\ 2\,y_{0i}(2\,y_i^1 + y_i^0 - 2\,\Delta t y_{1i}) + (y_i^1 2\,\Delta t y_{1i})^2) &= f_i^n + \textstyle\sum_{j=1}^M \delta(x - a_j)\delta v_j^n \end{aligned}$$

[2] v. [1] p. 164.

The formula for y^{n+1} is a polynomeal of dregre 3 and it is derived from the schema given by Dean, Glowinski , *et.al.* remarks 6.1, p. 206 on [2]:

$$\frac{\Phi(\zeta_1)-\Phi(\zeta_2)}{\zeta_1-\zeta_1}=\phi(\zeta_1) \ \text{ if } \ ||\zeta_1|| \text{ is small}$$

where $\Phi'=\phi$.
In fact, we apply it again for solve y_i^{n+1} for $n=1,\ldots,N$

$$\begin{aligned}
y_i^0 &= y_0(x_i)\; i=0,\ldots,N\\
\frac{(y_i^1)^3}{8}+\frac{y_i^0-\Delta t y_{1i}}{8}(y_i^1)^2+(\frac{(y_i^0-\Delta t y_{1i})^2}{8}+\frac{(\Delta t y_{1i})^2}{8}+\frac{2}{(\Delta t)^2})y_i^1 &+\\
-\frac{2\,\Delta t\, y_{1i}+2\,y_i^0}{(\Delta t)^2}-\frac{y_{i+1}^0-2\,y_i^0+y_{i-1}^0}{(\Delta x)^2}
\end{aligned}$$

$$a_n\,(y^{n+1})^3+b_n\,(y^{n+1})^2+c_n\,(y^{n+1})^3+d_n\,(y^{n+1})^3=0;$$

3 Comments

References

[1] R. Glowinski, and R. J.L. Lions Exact and approximatte controllability for distributed parameter systems, *Acta Numerical*, pp. 159–333, 1996.

[2] E.J. Dean R. Glowinski, Y.M Kuo and M. G. Nasser On The Discretization Of Some Second Order in Time Differential Equations Applications To Nonlinear Wave Problems ,*The Proceedings Of The NASA-UCLA Workshop On Computational Thechniques In Identification And Control Of Flexible Flight Structures*, November 2–4, Ed. A.V. Balakrishnan, pp. 199–246, 1989.