# Classical and Quantum Algorithms for the Boolean Satisfiability Problem

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**Abstract**

This paper presents a complete algorithmic study of the decision Boolean Satisfiability Problem under the classical computation and quantum computation theories. The paper depicts deterministic and probabilistic algorithms, propositions of their properties and the main result is that the problem has not an efficient algorithm (NP is not P). Novel quantum algorithms and propositions depict that the complexity by quantum computation approach for solving the Boolean Satisfiability Problem or any NP problem is lineal time.

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## 1 Introduction

The complexity of algorithms have the classification NP-Soft $\preccurlyeq$ NP-Hard. The problems in NP (Soft or Hard) are in a classes, with two characteristics they have a verification algorithm with polynomial complexity, and any problem in NP can be translated between them.

There is no a book about algorithms, complexity or theory of computation that it has not the description of the Boolean Satisfiability Problem, named SAT. It is a classical problem and one of the first to be in NP-Soft.

A Boolean variable only takes the values: 0 (false) or 1 (true). The logical operators are **not**: $\overline{x}$; **and**: $\wedge$, and **or**: $\vee$.

A SAT problem is a system Boolean formulas in conjunctive normal form over Boolean variables. Hereafter SAT$(n, m)$ is a problem with $n$ Boolean variables, and $m$ row Boolean formulas with at least one Boolean variable. By example, SAT$(4, 3)$:

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$$\begin{array}{lllll} & (x_3 & \vee \overline{x}_2 & & \vee x_0) \\ \wedge & (x_3 & \vee x_2 & \vee x_1) & \\ \wedge & ( & \overline{x}_2 & \vee x_1 & \vee x_0) \\ \wedge & ( & & & x_0). \end{array}$$

The assignation $x_0 = 1, x_1 = 0, x_2 = 1$, and $x_3 = 1$ is a solution because, such values satisfied the previous SAT$(4,3)$ but the assignation $x_0 = 0, x_1 = 0, x_2 = 0, x_3 = 0$ does not satisfied it.

The problem is to determine, does SAT$(n,m)$ have a solution? The answer yes or no is the decision problem. Being skeptical, also a specific solution is needed, the complexity of the verification is polynomial.

Any SAT$(n,m)$ can be translated into a set of ternary numbers. For this $\Sigma = \{0,1,2\}$, and each row of SAT$(n,m)$ maps to a ternary number, with the convention: $\overline{x}_i$ to 0 (false), $x_i$ to 1 ( true), and 2 when the variable $x_i$ is no present.

For example SAT$(5,3)$:

$$\begin{array}{llllll} & ( x_4 & \vee \overline{x}_3 & & & \vee x_0) \\ \wedge & ( & x_3 & \vee x_2 & & ) \\ \wedge & ( \overline{x}_4 & \vee x_3 & \vee \overline{x}_2 & \vee x_1 & ). \end{array}$$

It is traduced to:

10221
21122
01012.

The assignment $x_0 = 1$, $x_3 = 1$ satisfies the previous SAT$(5,3)$. It is not unique. Under this formulation, the search space of SAT is $[0, 3^n - 1]$ where $n$ is the number of Boolean variables.

Moreover, reciprocally the number 21221 can be represented as a Boolean formula. The previous SAT$(5,3)$ is modified as SAT$(5,4)$:

$$\begin{array}{llllll} & ( x_4 & \vee \overline{x}_3 & & & \vee x_0) \\ \wedge & ( & x_3 & \vee x_2 & & ) \\ \wedge & (\overline{x}_4 & \vee x_3 & \vee \overline{x}_2 & \vee x_1 & ) \\ \wedge & ( & x_3 & & & \vee x_0). \end{array}$$

where 21221 is translated into the last formula. Here, the system is like a fixed point system, where 21221 is a solution into the set of numbers of SAT$(5,4)$. This point out that to look for a solution is convenient to star with the numbers of SAT$(n,m)$, and then continues with the numbers in $[0, 3^n - 1]$.

The size of research space of SAT$(n,m)$ Boolean variables under the ternary translation is $3^n$, this means: 1) the number of different Boolean equations $(m)$ could be from 1 to $3^n$, 2) It is drawback to analyze the row formulas in order to determine properties or to rearrange their equations when $m$ is large. For $m \approx 3^n$, $m$ is an exponential factor to take in consideration to build knowledge for SAT$(n,m)$.

On the other hand, SAT$(n, m)$ can be studied as a fixed point system, and there is a simple subproblem with equal Boolean formulas size SSAT$(n, m)$where the translated formulas is onto $\{0, 1\}$. It has the nice property to replace the huge search space from $[0,\, 3^n - 1]$ to $[0,\, 2^n - 1]$.

Hereafter, the simple SAT is denoted by SSAT$(n, m)$ where $n$ is the number of Boolean variables and $m$ is the number of rows the problem with $n$ Boolean variables in each row .

An equivalent visual formulation is with ■ as 0, and □ as 1. Each row of a SSAT$(n, m)$ uses the $n$ Boolean variables to create a board, i.e. there is not missing variables. The following boards have not a set of values in $\Sigma$ to satisfy them:

| $x_1$ |
|---|
| ■ |
| □ |

| $x_2$ | $x_1$ |
|---|---|
| ■ | ■ |
| □ | □ |
| □ | ■ |
| ■ | □ |

I called unsatisfactory boards to the previous ones. It is clear that they have not a solution because each number has its binary complement, i.e., for each row image there is its complement image. By example, the third row (□■) and fourth row (■□) are complementary.

To find an unsatisfactory board is like order the number and its complement, by example: 000, 101, 110, 001, 010, 111, 011, and 100 correspond to the unsatisfactory board:

000
111
001
110
010
101
011
100.

An algorithm to solve SSAT$(n, m)$ by building an unsatisfactory board is:

**Algorithm 1.** ***Input:*** *SSAT$(n, m)$.*

***Output:*** *The answer if SSAT$(n, m)$ has solution or not. $T$ is an unsatisfactory board when SSAT$(n, m)$ has not solution.*

***Variables in memory:*** *$T[0 : 2^n - 1] = -1$: array of binary integer; address: integer; ct $= 0$: integer; k: binary integer.*

1. ***while not end(****SSAT$(n, m)$****)***
2. $k = b_{n-1}b_{n-2}\ldots b_0 =$ ***Translate to binary formula*** *(SSAT$(n, m)$);*
3. ***if*** $k.[b_{n-1}]$ ***equal*** $0$ ***then***

$address = 2 * k.[b_{n-2} \dots b_0]$;

***else***

$address = 2 * (2^{n-1} - k.[b_{n-2} \dots b_0]) - 1$;

***end if***

***if*** *T[address]* ***equal*** $-1$ ***then***

$ct = ct + 1$;

$T[address] = k$;

***end if***

***if*** $ct$ ***equal*** $2^n$ ***then***

***output:*** *There is not solution for* $SSAT(n, m)$.

***stop***

***end if***

***end while***

***output:*** $SSAT(n, m)$ *has a solution.*

When SSAT$(n, m)$ has not solution, table T is the witness because it is an unsatisfactory board. On the other hand, when SSAT$(n, m)$ has solution the algorithm testify that exists a solution. The complexity of the previous algorithm is $\mathbf{O}(m)$. No matters if the rows of SSAT$(n, m)$ are many or duplicates or disordered, however it could finished around $2^n$ iterations.

It will take many unnecessary steps to use a sorting procedure but Address Calculation Sorting (R. Singleton, 1956) [5]. To use a sorting procedure or to look for duplicate rows are not convenient when $m \approx 2^n$. Singletons sorting uses direct addressing into the search space $[0, 2^n - 1]$, and keeps the complexity to the numbers of rows of SSAT$(n, m)$.

The resulting board contains in the odd rows $x = 0b_n \dots b_0$ and in the next row $\overline{x}$. A solution of SSAT$(n, m)$ is the first address of the rows of $T$ where there is not a pair $x$ and $\overline{x}$.

In order to look for the solution of SSAT$(n, m)$, no previous knowledge is assumed. It is no necessary to set in order the rows of SSAT$(n, m)$, this algorithm determines for each row its address by addressing directly to its board position. The visual inspection of $T$ point out a solution of SSAT$(n, m)$, however a particular solution could be difficult to find by inspection when $m$ is very large.

The algorithm 2 in the next section provides the answer and the solutions with the similar complexity.

**Proposition 1.** *Let SSAT$(n, 2^n)$ be where formulas as the squares correspond to the* $0$ *to* $2^n - 1$ *binary numbers. Then it is an unsatisfactory board.*

*Proof.* The binary strings of the values from 0 to $2^l - 1$ are all possible assignation of values for the board. These strings correspond to all combinations of $\Sigma^n$. It means that for any possible assignation, there is the opposite Boolean formula with value 0 and therefore SSAT$(n, 2^n)$ has not solution. □

**Proposition 2.** *Given SAT$(n, m)$. There is not solution, if $L$ exists, where $L$ is any subset of rows, such that they are isomorphic to an unsatisfactory board.*

*Proof.* The subset $L$ satisfies the proposition 1. Therefore, it is not possible to find satisfactory set of $n$ values for SAT$(n, m)$. □

Here, the last proposition depicts a necessary condition in order to determine the existence of the solution for SAT. It is easy to understand but it is highly complicated to determine the existence of an unsatisfactory board in the general SAT. However, this point out to focus studying the simple SAT with Boolean formulas where each row has the same number of variables, i.e., SSAT$(n, m)$.

Hereafter, a more simply version of SAT is studied to map into binary strings with the alphabet $\Sigma = \{0, 1\}$. Here, each row of SSAT$(n, m)$ uses all the $n$ Boolean variables. This allows to define the binary number problem of the translated row formulas. The binary number problem consists to find a binary number from 0 to $2^n - 1$ that has not its binary complement into the translated values of the formulas of the given SSAT$(n, m)$. When the formulas of the given SSAT$(n, m)$ are different and $m = 2^n$ by the proposition 1 there is not satisfactory assignment.

**Proposition 3.** *Let SSAT$(n, m)$ be with different row formulas, and $m < 2^n$.*

*There is a satisfactory assignation that correspond to a binary string in $\Sigma^n$ as a number from* $0$ *to* $2^n - 1$.

*Proof.* Let $s$ be any binary string that corresponds to a binary number from 0 to $2^l - 1$, where $s$ has not a complement into the translated formulas. Then $s$ coincide with at least one binary digit of each binary number of the translated row formula. Therefore, each row of the SSAT$(n, m)$ is true.

□

The previous proposition explains when $s \in [0, 2^n - 1]$ exists for SSAT$(n, m)$. Also, proposition 1 states that if $m = 2^n$ and SSAT$(n, m)$has different rows, then there is not a solution. These are necessary conditions for any SSAT$(n, m)$.

The complexity to determine such $s$ is depicted in the next section.

## 2 SSAT$(n, m)$ has not properties for an efficient deterministic algorithm

This section is devoted to SSAT$(n, m)$ with each row formula has $n$ Boolean variables in descent order from $n-1$ to 0. This allows to translate each row to a unique binary string in $\Sigma^n$.

Recall, for SAT, I apply my technique: 1) to study general problem, 2) to determine a simple reduction, and 3) to analyze that there is no property for building an efficient algorithm for the simple problem.

On the other hand, the simple reduction provides a simple version of SAT, where for convenience all formulas have the same number of variables in a given order. In any case, this simple version of SAT is for studying the sections of complex SAT, and it is sufficient to prove that there is not polynomial time algorithm for it.

The situation to solve SSAT$(n, m)$ is subtle. Its number of rows could be exponential, but no more than $2^n$. It is possible to consider duplicate rows but this is not so important as to determine the set $\mathcal{S} \subset \Sigma^n$, where $\mathcal{S}$ is the set of the satisfactory assignations. On the other hand, $\Sigma^n$ is the search space for any SSAT$(n, m)$. It corresponds to a regular expression and it is easy to build it by a finite deterministic automata (Kleene's Theorem).

**Proposition 4.** *Let $\Sigma = 0, 1$ be an alphabet. Given a $SAT_{n\times m}$, the set $\mathcal{S} \subset \Sigma^n$ of the satisfactory assignations is a regular expression.*

*Proof.* $|\mathcal{S}|$ is a finite. □

It is trivial but from the computational point of view, the construction and translation of strings associate to SSAT$(n, m)$are easy to build.

Solving SSAT$(n, m)$could be easy if we have the binary numbers that has not a complement in its translated rows. Also, because, $|\Sigma^n| = 2^n$ has exponential size, it is convenient to focus in the information of SSAT$(n, m)$.

SSAT$(n, m)$ can be transformed in a fixed point formulation. This formulation is easy to understand.

1. An alphabet $\Sigma = \{0, 1\}$.
2. Each Boolean variable $x_i$ is mapped to 1, and $\overline{x}_j$ to 0.
3. A formula: $\overline{x}_{n-1} \vee \cdots x_l \vee \cdots \vee \overline{x}_1 \vee x_0$ can be transformed in its corresponding binary string in $\Sigma^n$; hereafter a binary string is considered by its value a binary number. By example $010 \in \Sigma^3$ is the binary number $10_2$.
4. If a $y \in \Sigma^n$, $\overline{y}$ is the complement binary string. This is done bit a bit where 0 is replaced by 1, and 1 by 0.

5. SSAT$(n,m)$ corresponds the set $M_{n\times m}$ of the $m$ binary strings of its formulas.

$$M_{n\times m} = \left\{ \begin{array}{c} s^1_{n-1}s^1_{n-2}\cdots s^1_1 s^1_0, \\ s^2_{n-1}s^2_{n-2}\cdots s^2_1 s^2_0, \\ \ldots, \\ s^m_{n-1}s^m_{n-2}\cdots s^m_1 s^m_0 \end{array} \right\}.$$

Note that the number $s^k_{n-1}\cdots s^k_1 s^k_0$ correspond to $k$ row of the SSAT$(n,m)$.

6. SSAT$(n,m)$ is a Boolean function. SSAT$(n,m)$: $\Sigma^n \to \Sigma$. The argument is a binary string of $n$ bits. An important consideration is that the complexity of the evaluation of this function is $\mathbf{O}(1)$. The figure 1 depicts SSAT$(n,m)$ as its logic circuit.

7. The decision problem SAT is equivalent to determine if SSAT$(n,m)$ is a blocked board or if exits $s \in [0,2^n]$ such that SSAT$(n,m)(s) = 1$.

The complexity of the evaluation of SSAT$(n,m)(y = y_{n-1}y_{n-2}\cdots\ y_1\ y_0)$ could be considered to be $\mathbf{O}(1)$. Instead of using a cycle, it is plausible to consider that SSAT$(n,m)$is a circuit of logical gates. This is depicted in figures 1 and 4. Hereafter, SSAT$(n,m)$ correspond to a logic circuit of "and", "or" gates, and the complexity of the evaluation of the SAT is $\mathbf{O}(1)$. The reason of this is to evaluate the SAT in an appropriate time, one step, no matters if $m \approx 2^n$.

**Proposition 5.**

*Let SSAT$(n,m)$ have different row formulas, and $m \leq 2^n$.*

1. *The complexity to solve SSAT$(n,m)$ is $\mathbf{O}(1)$.*

2. *Any subset of $\Sigma^n$ could be a solution for an appropriate SSAT$(n,m)$.*

*Proof.*

1. With the knowledge that the Boolean formulas are different, SSAT$(n,m)$ has solution when $m < 2^n$, i.e., it does not correspond to a blocked board. It has not solution when $m = 2^n$, i.e., it is a blocked board.

2. Any string of $\Sigma^n$ corresponds to a number in $[0,2^n-1]$.

   $\phi$ is the solution of a blocked board., i.e., for any SSAT$(n,m)$ with $m = 2^n$.

   For $m = 2^n - 1$, it is possible to build a SSAT$(n,m)$ with only $x$ as the solution. The blocked numbers $[0,2^n-1] \setminus \{x,\overline{x}\}$ and $x$ are copied to $\mathrm{M}_{n\times m}$. By construction, SSAT$(n,m)(x) = 1$, i.e., it is unique, and belongs to $\Sigma$.

   For $f$ different solutions. Let $x_1,\ldots,x_f$ be the given expected solutions. Build the set $C$ from the given solutions without any blocked pairs. Then the blocked numbers $[0,2^n] \setminus \{y \in \Sigma^n | x \in C, y = x \text{ or } y = \overline{x}\}$ and the numbers of $C$ are put in $\mathrm{M}_{n\times m}$.

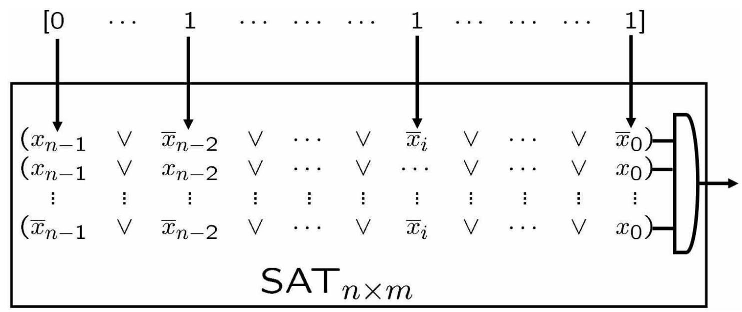


Figure 1: SSAT$(n, m)$ is a white of box containing a circuit of logical gates where each row has the same number of Boolean variables.

□

The proposition 5 depicts the importance of the knowledge. Here, "different rows formulas", it implies that solving SSAT$(n, m)$ is trivial. On the other hand, how much cost is to determine or build such knowledge. People build algorithms based in previous knowledge, many times without nothing at all.

The study of SSAT$(n, m)$ is without any previous assumptions, it is a white box with row Boolean formulas in conjunctive normal form. Under this point of view, it is not possible to know if the rows are different at first.

Let it be the set $\widehat{M}\left(\mathrm{SAT}_{n\times m}\right) = \{x \in \Sigma^n | \ \mathrm{SSAT}(n, m)(x) = 0\}$. It contains all the complement binary string that they are no solution of SSAT$(n, m)$.

The complement of a set is defined as usual, and it is denoted by $^c$.

$\widehat{M}\left(\mathrm{SAT}_{n\times m}\right)^c = \{x \in \Sigma^n | x \notin \widehat{M}\left(\mathrm{SAT}_{n\times m}\right)\}$. The context universal set is $\Sigma^n$, and the empty set is denoted by $\phi$.

**Proposition 6.** *Any $x \in \widehat{M}\left(SAT_{n\times m}\right)^c$ is satisfactory assignment of SSAT$(n, m)$.*

*Proof.* It is immediately, $\widehat{M}\left(\mathrm{SAT}_{n\times m}\right)^c = \{x \in \Sigma^n | \mathrm{SAT}_{n\times m}(x) = 1\}$. □

The numerical formulation provides immediately a necessary condition for the existence of the solution of SSAT$(n, m)$: the set $\widehat{M}\left(\mathrm{SAT}_{n\times m}\right)^c$ must be not empty. Here, we have three numerical binary sets $M_{n\times m}$, $\widehat{M}\left(\mathrm{SAT}_{n\times m}\right)$ and $\widehat{M}\left(\mathrm{SAT}_{n\times m}\right)^c$.

**Proposition 7.** *Given SSAT$(n, m)$. If $M_{n\times m} \cap \widehat{M}\left(SAT_{n\times m}\right)^c \neq \phi$. Then SSAT$(n, m)$ is like fixed point function, i.e., $\forall x \in M_{n\times m} \cap \widehat{M}\left(SAT_{n\times m}\right)^c$, $x \in M_{n\times m}$. On the other hand, If $M_{n\times m} \cap \widehat{M}\left(SAT_{n\times m}\right)^c = \phi$, then for a $x \in \widehat{M}\left(SAT_{n\times m}\right)^c$, $M_{n\times m} \cup \{x\}$ can be transformed into SSAT$(n, m)$, adding the translation of $x$ as a row formula in SSAT$(n, m)$.*

It is very expensive to analyze more than one time all the formulas of SSAT$(n, m)$. When $m \approx 2^n$, any strategy for looking properties has $m$ as a factor.

**Proposition 8.** *If $y \in \Sigma^n$, $y = y_{n-1}y_{n-2}\cdots y_1y_0$, then following strategies of resolution of SSAT$(n, m)$ are equivalent.*

1. *The evaluation of SSAT$(n, m)(y)$ as logic circuit.*

2. *A matching procedure that consists verifying that each $y_i$ match at least one digit $s_i^k \in M_{n\times m}$, $\forall k = 1, \ldots, m$.*

*Proof.* SSAT$(n, m)(y) = 1$, it means that at least one variable of each row is 1, i.e., each $y_i$, $i = 1, \ldots, n$ for at least one bit, this matches to 1 in $s_j^k$, $k = 1, \ldots, m$. □

The evaluation strategies are equivalent but the computational cost is not. The strategy 2 implies at least $m \cdot n$ iterations. This is a case for using each step of a cycle to analyze each variable in a row formulas or to count how many times a Boolean variable is used.

**Proposition 9.** *An equivalent formulation of SSAT$(n, m)$ is to look for a binary number $x^*$ from 0 to $2^n - 1$.*

1. *If $x^* \in M_{n\times m}$ and $\overline{x}^* \notin M_{n\times m}$ then SSAT$(n, m)(x^*) = 1$.*

2. *If $x^* \in M_{n\times m}$ and $\overline{x}^* \in M_{n\times m}$ then SSAT$(n, m)(x^*) = 0$. If $m < 2^n - 1$ then $\exists y^* \in [0, 2^n - 1]$ with $\overline{y}^* \notin M_{n\times m}$ and SSAT$(n, m)(y^*) = 1$.*

3. *if 2), then $\exists$ SSAT$(n, m + 1)$ such that 1) is fulfill.*

*Proof.*

1. When $x^* \in M_{n\times m}$ and $\overline{x}^* \notin M_{n\times m}$, this means that the corresponding formula of $x^*$ is not blocked and for each Boolean formula of SSAT$(n, m)(x^*)$ at least one Boolean variable coincides with one variable of $x^*$. Therefore SSAT$(n, m)(x^*) = 1$.

2. I have, $m < 2^n - 1$, then $\exists y^* \in [0, 2^n - 1]$ with $\overline{y}^* \notin M_{n\times m}$. Therefore, SSAT$(n, m)(y^*) = 1$.

3. Adding the corresponding formula of $y^*$, SSAT$(n, m + 1)$ is obtained. By 1, the case is proved.

□

This approach is quite forward for verifying and getting a solution for any SSAT$(n, m)$. By example, SSAT$(6, 4)$ corresponds to the set $M_{6\times 4}$:

| | $x_5 = 0$ | $x_4 = 0$ | $x_3 = 0$ | $x_2 = 0$ | $x_1 = 0$ | $x_0 = 0$ |
|---|---|---|---|---|---|---|
| | $\overline{x}_5\vee$ | $\overline{x}_4\vee$ | $\overline{x}_3\vee$ | $\overline{x}_2\vee$ | $\overline{x}_1\vee$ | $\overline{x}_0)$ |
| $\wedge($ | $\overline{x}_5\vee$ | $\overline{x}_4\vee$ | $\overline{x}_3\vee$ | $\overline{x}_2\vee$ | $\overline{x}_1\vee$ | $x_0)$ |
| $\wedge($ | $x_5\vee$ | $x_4\vee$ | $x_3\vee$ | $x_2\vee$ | $x_1\vee$ | $\overline{x}_0)$ |
| $\wedge($ | $\overline{x}_5\vee$ | $x_4\vee$ | $x_3\vee$ | $\overline{x}_2\vee$ | $x_1\vee$ | $x_0)$ |

| $x_5$ | $x_4$ | $x_3$ | $x_2$ | $x_1$ | $x_0$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |

.

The first table depicts that SSAT$(6,4)(y = 000000) = 1$. The second table depicts the set $M_{6\times4}$ as an array of binary numbers. The assignation $y$ corresponds to first row of $M_{6\times4}$. At least one digit of $y$ coincides with each number of $\mathrm{M}_{n\times m}$, the Boolean formulas of SAT$(6,4)$. Finally, $y = 000000$ can be interpreted as the satisfied assignment $x_5 = 0$, $x_4 = 0$, $x_3 = 0$, $x_2 = 0$, $x_1 = 0$, and $x_0 = 0$.

**Proposition 10.** *Given a SSAT$(n,m)$, there is a binary number $y \in M_{n\times m}$ such that $\overline{y} \notin M_{n\times m}$ then $y$ is fixed point for SSAT$(n,m)$ or SSAT$(n,m+1)(y)$, where SSAT$(n,m+1)$ is SSAT$(n,m)$ with adding the corresponding formula of $y$.*

*Proof.* This result follows from the previous proposition. □

SSAT$(n,m)$can be used as an array of $m$ indexed Boolean formulas. In fact, the previous proposition gives an interpretation of the SSAT$(n,m)$ as a type fixed point problem. For convenience, without before exploring the formulas the SAT, my strategy is to look each formula, and to keep information in a Boolean array of the formulas of SAT by its binary number as an index for the array. At this point, the resolution SSAT$(n,m)$ is equivalent to look for a binary number $x$ such that SSAT$(n,m)(x) = 1$. The strategy is to use the binary number representation of the formulas of SSAT$(n,m)$ in $\mathrm{M}_{n\times m}$.

A computable algorithm for solving SSAT$(n,m)$ is:

**Algorithm 2.** ***Input:*** *SSAT$(n,m)$.*

***Output:*** *$k$ : integer, such that SSAT$(n,m)(k) = 1$ or SSAT$(n,m+1)(k) = 1$ or SSAT$(n,m)$ has not solution.*

***Variables in memory:*** *$T[0 : 2^n - 1]$: array of double linked structure previous, next: integer; ct:=0 : integer; $first = 0$: integer; $last = 2^n - 1$: integer;*

1. ***while not end(****SSAT$(n,m)$****)***
2. $\quad k =$ ***Translate to binary formula*** *(SSAT$(n,m)$);*
3. $\quad$ ***if*** $T[k].next$ ***not equal*** $-1$ ***then***

```
        if SSAT(n, m)(k) equal 1 then
            output: k is a solution for SSAT(n, m);
            stop;
        else % Update the links of T
            T[T[k].previous].next=T[k].next;
            T[T[k].next].previous = T[k].previous;
            if k equal first then
                first := T[k].next;
            end if
            if k equal last then
                last := T[k].previous;
            end if
            T[k].next=−1;
            T[k].previous =−1;
            T[T[k̄].previous].next=T[k̄].next;
            T[T[k̄].next].previous = T[k̄].previous;
            if k̄ equal first then
                first := T[k̄].next;
            end if
            if k̄ equal last then
                last := T[k̄].previous;
            end if
            T[k̄].next=−1;
            T[k̄].previous =−1;
            ct := ct +2;
        end if
    end if
    if ct equal 2^n then
```

32. **output:** *There is not solution for SSAT*$(n, m)$*;*
33. **stop**;
34. **end if**
35. **end while**
36. $k = first$;
37. **output:** $k$ *a the solution for SSAT*$(n, m + 1)$ *adding the corresponding Boolean formula* $k$*;*
38. **stop**;

It is not necessary to use $m$, the algorithm works while there is a row to analyze.

Line 37 changes the input SSAT$(n, m)$ to SSAT$(n, m+1)$. This is for making SSAT$(n, m + 1)$ as point fixed problem, i.e., if the algorithm runs again, the solutions is already a formula in SSAT$(n, m + 1)$.

$T$ is an special vector array of links. It has a memory cost of $2^{n+1}$. Updating the links has a low fixed cost as it is depicted in lines 8 to 27. The array $T$ allows to design the previous algorithm in efficient way, to computes all the solutions, but the draw back is the memory cost.

Assuming that SSAT$(n, m)$ is a logic circuit (see remark 2) the complexity of its evaluation is $\mathbf{O}(1)$. This is a non trivial assumption that encloses that the Boolean formulas of a given SSAT$(n, m)$ are given but not analyzed previously. In fact, the formulas of the given problem could be repeated or disordered.

If a solution is not in $\mathrm{M}_{n\times m}$, the complexity of the previous algorithm is a cycle of size $m$ plus 1, i.e., $\mathbf{O}(m)$. But the worst case scenario is when $m \approx 2^n$, i.e., $\mathbf{O}(m) = \mathbf{O}(2^n)$. If the Boolean formulas are in separated of their complement, the complexity is $\mathbf{O}(m/2)$. Nevertheless, with duplicated formulas and without solution the algorithm is computable, i.e., the number of iteration is always bounded, with no duplicated formulas, it is at most $2^n$.

For the modified SSAT$(n, m + 1)$, inverting the cycle "for", i.e., **for** i:= m+1 **downto** 1, the complexity is $\mathbf{O}(1)$. The knowledge pays off but it is not a property to exploit to reduce the complexity for any SSAT$(n, m)$. It is a posteriori property to build SSAT$(n, m + 1)$ and to take advantage of it as a fixed point type of problem.

The next algorithm computes, in similar way, the set $\mathcal{S} \subset \Sigma^n$ of the solutions, i.e., the knowledge of SSAT$(n, m)$. When $|\mathcal{S}|$ is small or zero, to determine all the solutions or no solution has the same complexity $\mathbf{O}(2^n)$.

**Algorithm 3.** ***Input:*** *SSAT*$(n, m)$.

***Output:*** $T$ : *List of binary numbers such that,* $x \in T$*, SSAT*$(n, m)$*(x)=1, or SSAT*$(n, m)$ *has not solution and* $T$ *is empty.*

***Variables in memory:*** $T[0 : 2^{n-1}]$*: list as an array of integers, double link structure previous, next : integer; ct:=*$0$ *: integer;* $first = 0$*: integer;* $last = 2^n - 1$*: integer;*

```
while not end(SSAT(n,m))
    k = Translate to binary formula (SSAT(n,m));
    if T[k].previous not equal −1  or  T[k].next not equal −1 then
        if SSAT(n,m)(k) not equal 1 then % Update the links of T
            T[T[k].previous].next = T[k].next;
            T[T[k].next].previous = T[k].previous;
            if k equal first then
                first := T[k].next;
            end if
            if k equal last then
                last := T[k].previous;
            end if
            T[k].next = −1;
            T[k].previous =−1;
            T[T[k̄].previous].next = T[k̄].next;
            T[T[k̄].next].previous = T[k̄].previous;
            if k̄ equal first then
                first := T[k̄].next;
            end if
            if k̄ equal last then
                last := T[k̄].previous;
            end if
            T[k̄].next = −1;
            T[k̄].previous = −1;
            ct := ct + 2;
        end if
    end if
    if ct equal 2^n then
```

```
        output: There is not solution for SAT_{n×m};
        stop;
    end if
end while
stop
```

It is not necessary to use $m$, the algorithm works while there is a row to analyze.

For the extreme cases, when $n$ is large and $m \approx 2^n$ with no duplicates formulas, the cost of solving SSAT$(n, m)$ or building $T$ is similar, $\mathbf{O}(m)$. There is a high probability that without any knowledge of the positions of the formulas, the algorithm 2 executes $m$ steps.

At the end of the previous algorithm, the knowledge of SSAT$(n, m)$is the set $\mathcal{S}$. Before of exploring SSAT$(n, m)$, there are not plausible binary numbers associated with its solution in one step.

Now, let $\mathbb{K}_{n\times m}$ be knowledge of the solutions.

**Proposition 11.** *Given SSAT$(n, m)$.*

1. $\mathbb{K}_{n\times m} = \widehat{M}\left(SAT_{n\times m}\right)^c = \mathcal{S}$.
2. *The cost of building* $\mathbb{K}_{n\times m}$ *is* $\mathbf{O}(m)$.
3. *It is* $\mathbf{O} = 1$ *by solving SSAT$(n, m+1)$ from* $\mathbb{K}_{n\times m}$.

*Proof.*

1. The algorithm 3 builds $\widehat{M}\left(\mathrm{SAT}_{n\times m}\right)^c = \mathbb{K}_{n\times m}$. Also, by definition, $\mathcal{S} = \widehat{M}\left(\mathrm{SAT}_{n\times m}\right)^c$.
2. The algorithm 3 has only a cycle of size $m$.
3. Any $s \in \widehat{M}\left(\mathrm{SAT}_{n\times m}\right)^c$ solves SSAT$(n, m+1)$ in one iteration.

□

The proposition depicts the condition for solving efficiently SSAT$(n, m)$, which is $\mathbb{K}_{n\times m}$ the knowledge associated with the specific given SSAT$(n, m)$. After analyzing the formulas of SSAT$(n, m)$, we have $\widehat{M}\left(\mathrm{SAT}(n, m)\right)^c = \mathbb{K}_{n\times m}$ but no before.

The two previous algorithm use the array $T$. It combines index array and double links, the drawback is the amount of memory needed to update the links, but its cost is $\mathbf{O}(1)$.

The next tables depict that inserting and erasing binary numbers into $T$ (this structure correspond to $\mathcal{S}$) only consists on updating links with fixed complexity $\mathbf{O}(8)$ (number of the link assignations).

| $first = 0$; $last = 7$; | | |
|---|---|---|
| $T$ | | |
| $i$ | $previous$ | $next$ |
| 0 | $-1$ | 1 |
| 1 | 0 | 2 |
| 2 | 1 | 3 |
| 3 | 2 | 4 |
| 4 | 3 | 5 |
| 5 | 4 | 6 |
| 6 | 5 | 7 |
| 7 | 6 | $-1$ |

By example, let SSAT$(3, 2)$ be

$$\begin{array}{cccc} & ( x_2 & \vee \overline{x}_1 & \vee x_0) \\ \wedge & ( x_2 & \vee \overline{x}_1 & \vee \overline{x}_0). \end{array}$$

The resulting array $T$ with $\mathbb{K}_{3\times 2}$ after executing algorithm 3 is

| $first = 0$; $last = 7$; | | |
|---|---|---|
| $T$ | | |
| $i$ | $previous$ | $next$ |
| 0 | $-1$ | 1 |
| 1 | 0 | 4 |
| 2 | $-1$ | $-1$ |
| 3 | $-1$ | $-1$ |
| 4 | 1 | 5 |
| 5 | 4 | 6 |
| 6 | 5 | 7 |
| 7 | 6 | $-1$ |

The knowledge for a given SSAT$(n, m)$ depends of exploring it. Before, to explore SSAT$(n, m)$ there are not properties nor binary numbers associated with $\mathbb{K}_{n\times m}$. By properties, I mean, the algorithm 3 computes $\mathcal{S}$, now it is possible to state the properties of the all number in $\mathcal{S}$, by example, they could bye twin prime numbers.

Having $\mathbb{K}_{n\times m}$, the knowledge of SSAT$(n, m)$ for any numbers $x$ of the list $T$, the complexity for verifying SSAT$(n, m)(x) = 1$ is $\mathbf{O}(1)$.

It is trivial to solve SAT when knowledge is given or can be created in an efficient way. This results of this section are related to my article [2], where I stated that NP problems need to look for its solution in an search space using a Turing Machine for General Assign Problem of size $n$ (GAP$n$) : "It is a TM the appropriate computational model for a simple algorithm to explore at full the GAP$n$'s research space or a reduced research space of it". As trivial as it sound, pickup a solution for SSAT$(n, m)$ depends of exploring its $m$ Boolean formulas.

The question if there exists an efficient algorithm for any SSAT$(n, m)$ now can be answered. The algorithm 3 is technologically implausible because the amount of memory needed. However, building $\mathbb{K}_{n\times m}$ provides a very efficient telephone algorithm (see proposition 8, and remarks 5 in [1]) where succeed is guaranteed for any $s \in \mathbb{K}_{n\times m}$. On the other hand, following [1]) there are tree possibilities exhaustive algorithm (exploring all the searching space), scout algorithm (previous knowledge or heuristic facilities for searching in the search space), and wizard algorithm (using necessary and sufficient properties of the problem).

The study of NP problems depicts that only for an special type of problem exists properties allowing to build an ad-hoc efficient algorithms, but these properties can not be generalized for any GAP or any member of class NP.

The Boolean formulas of SSAT$(n, m)$ correspond to binary numbers in disorder (assuming an order of the Boolean variables as binary digits). The algorithm 2 has a complexity of the size of $m$ (the numbers of formulas of SAT).It is not worth to considered sorting algorithm because complexity increase by an exponential factor $m \approx 2^n$) when $m$ is very large.

Without exploring and doing nothing, for any SSAT$(n, m)$, the set $\mathrm{M}_{n\times m}$ can be an interpretation without the cost of translation. The associated $\mathrm{M}_{n\times m}$ can be consider an arbitrary set of numbers. The numbers in $\mathrm{M}_{n\times m}$ have not relation or property between them to point out what are the satisfied binary number in $\Sigma^n$. The numbers of $\mathrm{M}_{n\times m}$, as binary strings have the property to belong to the translation of SSAT$(n, m)$ but $\mathcal{S}\cap$ $\mathrm{M}_{n\times m} = \phi$.

In fact one or many numbers could be solution or not one, but also it is not easy to pick up the solution in the set $[0, 2^n - 1]$ without knowledge.

Hereafter, the Boolean formulas of SSAT$(n, m)$ are considered a translation to binary numbers in disorder and without any correlation between them.

**Proposition 12.** *With algorithm 2:*

1. *A solution for SSAT$(n, m)$ is efficient when $m << 2^n$.*
2. *A solution for SSAT$(n, m)$ is not efficient when $m \approx 2^n$.*

*Proof.* It follows from algorithm 2. □

**Proposition 13.** *The complexity to determine if SSAT$(n, m)$has solution, or to build the knowledge or to find the set $\mathcal{S}$ of SSAT$(n, m)$is the same and it is around $\mathbf{O}(m)$.*

*Proof.* It follows from algorithms 1, 2, and 3. □

The previous proposition states the complexity of the deterministic way to solve SSAT$(n, m)$without no prior knowledge. It is important to note that there are not iterations or previous steps to study SSAT$(n, m)$. Any of the algorithms 1, 2, and 3 face the problem without any assumptions of what formulas, or order, or structure could have it. For they, it is a like a circuit in a white box as a file of logic formulas.

The next section studies SSAT$(n, m)$ from the probabilistic point of view.

# 3 Probabilistic algorithm for SSAT$(n, m)$

In this section the research space $[0, 2^n - 1]$ is considered as a collection of objects with the same probability to be selected. Hereafter, $\mathcal{P}$ stands for a probability function.

**Proposition 14.** *Let SSAT$(n, m)$ have n large with no duplicates rows.*

1. $x \in [0, 2^n - 1]$ *has the same priori probability to be selected, i.e, uniform probability.*
2. $\mathcal{P}(\mathcal{S}) = |\mathcal{S}|/2^n$ *is fixed.*

*Proof.*
By the proposition 5 any arbitrary set $\mathcal{S} \subset \Sigma^n$ ($|\Sigma^n| = 2^n$) could be the set of solutions of an appropriate SSAT$(n, m)$.

1. Without previous knowledge or reviewing the formulas of SSAT$(n, m)$ any number could be selected with $\mathcal{P}(x) = 1/2^n$.
2. Any $x \in \mathcal{S}$ is the solution and any $y \in \mathcal{S}^c$ is not. This is fixed for a given SSAT$(n, m)$. Therefore $1 = \mathcal{P}(\mathcal{S}) + \mathcal{P}(\mathcal{S}^c)$. Moreover, $\mathcal{P}(\mathcal{S}) = |\mathcal{S}|/2^n$.

□

The uniform probability for selecting any $x \in [0, 2^n - 1]$ is $\mathcal{P}(\{x\}) = 1/2^n$. SSAT$(n, m)$is solved using the properties:

1. SSAT$(n, m)$is a function. Its formulas are disordered.
2. The problem to solve is to determine if exist or not a binary number $x \in [0, 2^n - 1]$, such that SSAT$(n, m)(x) = 1$.
3. $\forall x \in [0, 2^n - 1]$, such that SSAT$(n, m)(x) = 0$ then SSAT$(n, m)(\overline{x}) = 0$.
4. $n$ and $m$ are large.
5. Without any previous knowledge, any $x \in [0, 2^n - 1]$ has the same priori probability to be selected, i.e, uniform probability.
6. $m$ is arbitrary large, including the case $m > 2^n$. This means, SSAT$(n, m)$ could have duplicate formulas.
7. $\mathcal{P}(\mathcal{S}) = |\mathcal{S}|/2^n$ is fixed.
8. After testing $f$ binary numbers $x_1, x_2, \ldots, x_f$, such that SSAT$(n, m)(x_i) = 0$, the failure set is $F = \{x_1, \ldots, x_f, \}$. The probability of the selection of the rest candidates for solving SSAT$(n, m)$slightly increase, i.e., the posteriori probability of the candidates for solving SSAT$(n, m)$ of any $x \in [0, 2^n - 1] \setminus F$ is equal to $1/\left(2^n - |F|\right) = 1/\left(2^n - f\right) \approx 1/2^n$.

**Proposition 15.** *Let $n$ be large.*

1. *When $m \approx 2^n$ or $m > 2^n$, the probability for selecting a solution after $f$ failures ($f << 2^n$) is insignificant.*
2. *A solution for $SSAT(n, m)$ is efficient when $m << 2^n$.*
3. *A solution for $SSAT(n, m)$ is not efficient when $m \approx 2^n$.*

*Proof.*

1. Let $P(\text{So}) = |\mathcal{S}|/2^{2n}$ be the probability of selecting a solution $s \in \mathcal{S}$, where $\mathcal{S}$ is the set of the solutions. Assuming $m$ huge, $\text{SSAT}(n, m)$ has many different rows (also, it is possible to have duplicates rows), therefore the set $|\mathcal{S}|$ is very small. Let $P(\text{Se}(f)) = 1/(2^n - f)$ be the probability after $f$ failures. Then the probability $P(\text{Se}(f) \cap \text{So}) = 1/(2^n - f) \cdot |\mathcal{S}|/2^n \approx |\mathcal{S}|/2^{2n} \approx 0$.
2. When $m \ll 2^n$, $\text{SSAT}(n, m)$ has a high probability that its rows are not blocked, i.e., $|\mathcal{S}| \approx 2^n$. Therefore, the probability for solving $\text{SSAT}(n, m)$, $P(\text{Se}) = |\mathcal{S}|/2^n$ is almost 1, i.e., many numbers in $[0, 2^n - 1]$ are solutions. It is fast to pick $x \in [0, 2^n - 1] \cap \mathcal{S}$.
3. In this case, there are only few numbers to solve $\text{SSAT}(n, m)$ many rows of $\text{SSAT}(n, m)$ are blocked, i.e., $|\mathcal{S}| \ll 2^n$. $P(\text{Se}(f) \cap \text{So}) \approx |\mathcal{S}|/2^{2n} \approx 0$. It is insignificant and take a long time to find a solution.

□

In order to find a solution for SAT, there are a probabilistic algorithms rather than the previous deterministic algorithms. By example, a probabilistic algorithm for $\text{SSAT}(n, m)$ is:

**Algorithm 4.** ***Input:*** *$SSAT(n, m)$.*
***Output:*** *$k$ : integer, such that $SSAT(n, m)(k) = 1$ or $SSAT(n, m+1)(k) = 1$ or $SSAT(n, m)$ is not satisfied and all entries of $T$ are 1.*
***Variables in memory:*** *$T[0 : 2^{n-1}] = 0$ of Boolean; ct:=0 : integer;*

1. ***while (1)***
2.     ***Select uniform randomly*** *$k \in [0, 2^n - 1] \setminus \{i \,|\, T[i] = 1\}$;*
3.     ***if*** *$SSAT(n, m)(k)$* ***equals*** *1* ***then***
4.         ***output:*** *$k$ is the solution for $SSAT(n, m)$.*
5.         ***stop***
6.     ***end if***
7.     ***if*** *$T[k]$* ***not equal*** *1* ***then***

8. $T[k] := 1$;

9. $ct := ct + 1$;

10. ***if*** $ct$ ***equal*** $2^n$ ***then***

11. ***output:*** *There is not solution for SSAT*$(n, m)$.

12. ***stop***

13. ***end if***

14. ***end while***

This algorithm is computable. It always finishes with the answer for any SSAT$(n, m)$. In each cycle, the probability of the selection of the candidates is slightly increased and one binary number is omitted from the list of candidates but the selection is still probabilistic uniform for the remain candidates. It is $1/(2^n - k)$ after $k$ cycles.

**Proposition 16.** *The algorithm 4:*

1. *is efficient for finding a solution for SSAT*$(n, m)$ *when* $m << 2^n$.
2. *is not efficient for finding a solution or to determine if SSAT*$(n, m)$*has a solution when* $m \approx 2^n$ *or* $m \geq 2^n$.

*Proof.*

1. When $m << 2^n$, it implies, $\mathcal{P}(\text{SSAT}(n, m)(k) = 1) \approx 1$ for many $k$. The algorithm is highly probable to solve SSAT$(n, m)$ in short time.
2. When $m \approx 2^n$, it is highly probable that many selected uniform randomly number in $[0, 2^n - 1]$ are blocked, i,e., $\mathcal{P}(\text{SSAT}(n, m)(k) = 0) \approx 1$ for many $k$ of the step 2. The algorithm 4 needs to test a huge amount of numbers for increasing the probability for selecting a solution. An extreme case is with only few binary numbers as the solution, but the worst case is when SSAT$(n, m)$ is a blocked board, in this case the algorithm executes $2^n$ steps. Even with duplicates formulas, i.e., when $m \geq 2^n$, the algorithm takes at less $2^n$ steps to determine no solution.

□

Finally,

**Proposition 17.** *SSAT*$(n, m)$ *has not property or heuristic to build an efficient algorithm.*

*Proof.* For any SSAT$(n, m)$, the translation of its formulas to binary numbers is implicit without cost by assuming the following implicit rules: $x_i$ is 1, and $\overline{x}_i$ is 0, and the order of the variables in each row is the same. $\mathrm{M}_{n\times m}$ is the set of binary numbers, it could be in disorder, and without any previous knowledge, also, there are not correlation between the binary strings numbers in it. The only property between number is SSAT$(n, m)(x) = 0$ if and only if SSAT$(n, m)(\overline{x}) = 0$. But, this property does not increase the probability for be the solution after $f \ll 2^n$ failures by prop 15.

If a property or a heuristic exists such that an arbitrary probabilistic algorithm quickly determines when SSAT$(n, m)$ has or not a solution then even in the extreme case (when there is not solution, i.e., when the formulas of SSAT$(n, m)$ correspond to a blocked board), the number of steps could at less $2^{n-1}$ for the algorithms 2 and 3 or $2^n$ for the algorithms 1 and 4.

For the safe of the argumentation let us considerer $n$ and $m$ large, $m \approx 2^n$ or $m \gg 2^n$. Also, the solution of SSAT$(n, m)$could be any binary number or none.

With $m \gg 2^n$, the deterministic algorithms like the algorithms 1, 2, and 3 are not efficient, $\mathbf{O}(2^n)$ .

If an algorithm similar to the algorithm 4 finds very quickly $x^* \in [0, 2^n - 1]$ such that SSAT$(n, m)(x^*) = 1$ then the selection of the candidates (step 2) has not uniform probability.

Using such property, any arbitrary subset of numbers must have it, otherwise, the algorithm is not efficient for any SSAT$(n, m)$. But, this property alters the prior probability of the uniform distribution. Such characteristic or property implies that any natural number is related to each other with no uniform probability at priory for be selected.

This means, that such property is in the intersection of all properties for all natural numbers. Also it is no related to the value, because, the intersection of natural classes under the modulo prime number is empty.

Moreover, it point outs efficiently to the solution of SSAT$(n, m)$, it means that this property is inherently to any number to make it not equally probable for its selection. Also, the binary numbers of $\mathrm{M}_{n\times m}$ correspond to an arbitrary arrangement of the Boolean variables of SSAT$(n, m)$, so in a different arrangement of the positions for the Boolean variables, any number has inherently not uniform probability for selection. On the other hand, for the extreme case when the formulas of SSAT$(n, m)$ correspond to a blocked board, the property implies that no solution exits with the numbers of steps $\ll 2^{n-1}$, i.e., the answer is found without reviewing all the formulas of SSAT$(n, m)$! □

It is an absurd that such property exits, because it changes the probability of the uniform distribution of similar objects without previous knowledge. Without an inspection of the formulas of SSAT$(n, m)$when $n \gg 1$, and $m = 2^n - 1$, the unique solution has $1/2^n$ as the priory probability of be the solution. After $k$ cycles, with the information of the failed candidates, the probability slightly grows to $1/(2^n - 2k)$, i.e., the probability of the uniform selection taking in consideration the failed candidates does not grow exponentially but lineally.

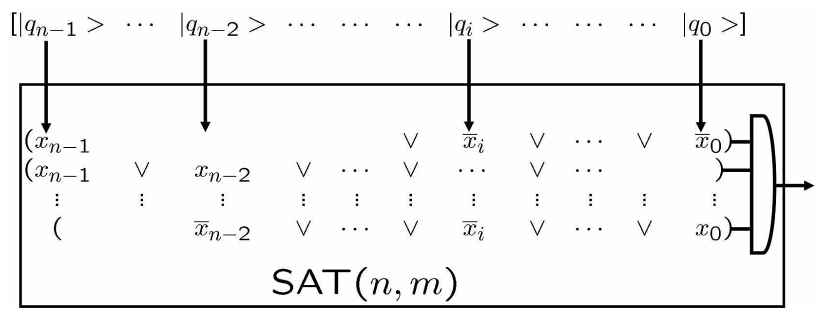


Figure 2: Quantum Boolean variables input SAT$(n, m)$ for the algorithm 5 at step 1.

Therefore, any probabilistic algorithm must take a lot of time to determine the solution of SSAT$(n, m)$.

**Proposition 18.** *NP has not property or heuristic to build an efficient algorithm.*

*Proof.* Let X be a problem in NP. $\text{PH}_\text{X}$ is the set of properties or heuristics for building an efficient algorithm for problem X.

$$\bigcap_{\text{X}} \text{PH}_\text{X} = \phi$$

by the previous proposition. □

**Proposition 19.** *A lower bound for the complexity of SSAT$(n, m)$for the extreme case when there is no solution or only one solution is $2^{n-1} - 1$. Therefore, $\mathbf{O}(2^{n-1}) \preccurlyeq$ NP-Soft $\preccurlyeq$ NP-Hard.*

*Proof.* For the case when there is only one solution, the probabilistic algorithm 4 after $k$ failed numbers has a probability equals $1/(2^n - k)$. The number of tries to get a probability equals to $1/2$ is $k = 2^n - 2$. On the other hand, with the algorithm 2 for the case when there is no solution the number of iterations could be at less $2^{n-1}$.

No matter if the algorithm is deterministic or probabilistic a lower bound for the complexity of SSAT$(n, m)$ is $\mathbf{O}(2^{n-1})$. This implies, $\mathbf{O}(2^{n-1}) \preccurlyeq$ NP-Soft $\preccurlyeq$ NP-Hard. □

# 4 Quantum computation

Grover's Algorithm can be adapted to look for the binary number that it solves SSAT$(n, m)$, adapting the function $C$ of the clauses of SAT as Grover depicts

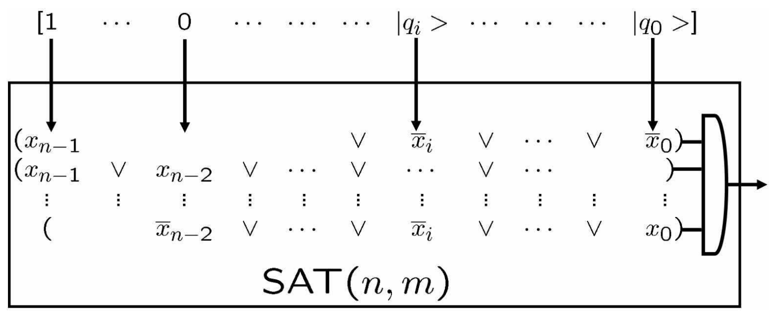


Figure 3: Boolean values, and quantum Boolean variables input SAT$(n, m)$ for the algorithm 5 at step 7.

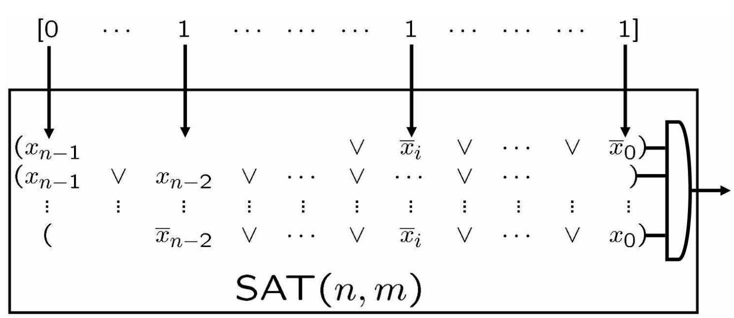


Figure 4: Final step of the algorithm 5 at step 11 with the solution for SAT$(n, m)$.

in [4]. Using this adaptation, the Grover's Algorithm is $\mathbf{O}(2^{n/2})$. This is not good but it is better than $\mathbf{O}(2^n)$. In [6], there is survey of the quantum potential and complexity and the Shor's algorithm for prime factorization and other discrete algorithms.

Here, the quantum computation approach is used to formulate a novel algorithm based on quantum hardware for the general SAT. Similar to the idea of Jozsa [1992] and Berthiaume and Brassard [1992, 1994] of a random number generator with a Feynman [3] reachable, and reversible quantum circuit approach.

The main idea is to adapt the algorithm 2 for using quantum variables for uniformly exploring $[0, 2^n - 1]$ states at the same time. The following proposition states how the quantum Boolean variables behave:

**Proposition 20.** *$n$ quantum Boolean Variables, uniformly explore $\{0,1\}^n$.*

*Proof.* If they do not. Then there is a priori property that it can explain why. But this means that they have not the random behavior of the quantum phenomenon related. □

In the algorithm 2, the cycle **for** i:=0 to $2^n - 1$ is replaced by the input of $n$ quantum binary variables. Instead of doing iterations, the coupling of the quantum variables with circuit of SSAT$(n, m)$ reaches its reversible and stable state in one operation. Figure 2 depicts quantum variables instead of the cycle **for** i:=0 to $2^n - 1$. This approach is based to the couple quantum hardware with a reversible quantum circuit of SAT. The only two outcomes 0 or 1 are found after the quantum circuit reach it stable and reversible state.

Here, the problem to solve is the general SAT, i.e., SAT$(n, m)$ where $n$ is the number of Boolean variables and its rows Boolean formulas could have from one to $n$ Boolean variables. The property to build this quantum hardware-software depends to couple the input quantum variables with the Boolean circuit of SAT$(n, m)$ for interacting to reach a reversible and stable solution after the quantum exploration of the search space. The assumption is that $n$ quantum variables have the $2^n$ Boolean values as input of SAT$(n, m)$. After a while or maybe instantaneous, the couple reach the stable and reversible state, then the output of SAT$(n, m)$ only have two outcomes. When the outcome is 0, there is not solution. Otherwise, a solution is computed bit by bit.

It is important to note that the complexity of solving SAT$(n, m)$ is independent of $m$, the number of Boolean formulas.

The following algorithm depicts how to solve efficiently SAT$(n, m)$ by quantum computation.

**Algorithm 5.**

***Input:*** *SAT$(n, m)$.*

***Output:*** *$x$ such that SAT$(n, m)(x) = 1$ or SAT$(n, m)$ has not solution, where $x[0 : n - 1]$ :* ***array of Boolean variables;***

***Variables in memory:*** *$q[0 : n - 1]$:* ***Array of quantum Boolean variables;*** *$v$:* ***integer.***

1. ***if*** $SAT(n,m)(q_n, q_{n-1}, \ldots, q_0)$ ***equal*** *0* ***then***
2. ***output:*** *There is not solution for* $SAT(n,m)$.
3. ***stop***
4. ***end if***
5. ***for*** $v = n-2$ ***downto*** $0$
6. $x_{v+1} = 0;$
7. ***if*** $SAT(n,m)(x_{n-1}, \ldots, x_{v+1}, q_v, \ldots, q_0)$ ***equals*** *0* ***then***
8. $x_{v+1} = 1;$
9. ***end if***
10. ***end for***
11. ***output:*** $x = [x_{n-1}, x_{n-1}, \ldots, x_0]$ *is the solution for* $SAT(n,m)$.
12. ***stop***

The first step is the answer to question if SAT has or not solution. The rest of the steps build a solution by uniformly exploring $\{0,1\}^v$, $v = n-2, \ldots, 0$ quantum Boolean variables. It is crucial to compute a solution, as a witness to verify by direct evaluation that such $x$ satisfies $\mathrm{SAT}(n,m)(x) = 1$.

Figure 3 depicts the substitution of the quantum variables by the corresponding Boolean values that satisfies $\mathrm{SAT}(n,m)$ at the step $i$. Here, there are not a tree of alternatives, there are only two choices for each Boolean variable when the the quantum variables are been substituting.

If the assumption of the coupling quantum variables with a $\mathrm{SAT}(n,m)$ works, then the complexity of the previous algorithm is $\mathbf{O}(1)$ to answer the decision SAT. But $\mathbf{O}(n)$ is for building a solution $x$. With $x$ the verification that $\mathrm{SAT}(n,m)(x) = 1$ is easy and straight forward.

Moreover, because of the equivalence of the class NP, any $\mathrm{SAT}(n,m)$ or NP problem under quantum computation has the same complexity! The article [2] depicts that the search space of GAP is finite and numerable, therefore a similar a coupling between quantum Boolean variables and the cycles can be used for any NP.

On the other hand, the next algorithm verifies no solution for SAT.

**Algorithm 6.** ***Input:*** $SAT(n,m)$.
***Output:*** $SAT(n,m)$*is or not consistent with no solution;*
***Variables in memory:*** $q[0:n-1]$: ***Array of quantum Boolean variables;*** $v$: ***integer.***

1. ***if*** $SAT(n,m)(q_n, q_{n-1}, \ldots, q_0)$ ***equal*** *0* ***then***
2. ***for*** $v = n-1$ ***downto*** $0$

*3.* **if** $SAT(n, m)(q_{n-1}, \ldots, 0, q_v, \ldots, q_0)$ ***equals*** *0* ***or***
$SAT(n, m)(q_{n-1}, \ldots, 1, q_v, \ldots, q_0)$ ***equals*** *1* ***then***

*4.* ***output:*** *No solution is inconsistent for* $SAT(n, m)$.

*5.* ***stop***

*6.* ***end if***

*7.* ***end for***

*8.* ***output:*** *No solution is consistent for* $SAT(n, m)$.

*9.* ***stop***

The previous algorithm has complexity $\mathbf{O}(n)$. It states an important proposition.

**Proposition 21.** *A quantum algorithm for uniformly exploring* $\{0, 1\}^n$ *can be only verified by a similar quantum algorithm in short time.*

*Proof.* It follows from the propositions of the previous section that quantum computational approach can review the search space $\{0, 1\}^n$ but traditional algorithms take exponential number of iterations, i.e.,$\mathbf{O}(2^{n-1})$. □

To my knowledge, under the quantum theory the outcome of the step 4 is impossible but human error or failure in the construction of the couple quantum hardware with the logic circuit.

# Conclusions and future work

Heuristic techniques, using previous knowledge do not provide reducibility (see 6 in [2]) for all the NP problems. The lack of a common property for defining an efficient algorithm took a long way. My research focused in the verification of the solutions in polynomial time. I point out that there is not an efficient verification algorithm for solving a NP Hard problem (GAP is a minimization problem). I develop a reduction method, here it is applied to the SAT.

The classical SAT (a NP decision problem), where formulas have any number of Boolean variables, is reduced to the simple version SSAT$(n, m)$. This allows to focus in the characteristics and properties for solving the simple SSAT$(n, m)$ with the reduction of the search space from $\{0, 1, 2\}^n$ to $\{0, 1\}^n$, and the number problem formulation for SSAT$(n, m)$. The proposition 17 states that there is not a property for building an efficient algorithm for SSAT$(n, m)$. It is immediately an efficient algorithm does not exist for any SAT or NP.

For the future, I am interesting in reviewing the computational models, languages and theory of computation under the quantum computational approach.

Quantum computation is opening new perceptions, paradigms and technological applications. Today or near future, it is possible no programming at all, for building a very complicate computational system will be to design and grow a huge complex crystalline structure of a huge massive quantum logic circuit. It is possible that the only way to test this type of design is by the quantum computational approach as it is stated by proposition 21.

Finally, SSAT$(n, m)$ supports and proves that there is not a property to reduce the complexity of the worst case for a decision NP problem, i.e., NP $\neq$ P. On the other hand, a coupling of quantum variables with SAT$(n, m)$ provides the novel algorithm 5, which it states: any NP problem has lineal complexity for its solution under an appropriate quantum computational design.

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