Lower bound for the Complexity of the Boolean Satisfiability Problem

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Abstract

This paper depicts algorithms for solving the decision Boolean Satisfiability Problem. An extreme problem is formulated to analyze the complexity of algorithms and the complexity for solving it. A novel and easy reformulation as a lottery for an extreme case is presented to determine a stable complexity around 2^n. The reformulation point out that the decision Boolean Satisfiability Problem can only be solved in exponential time. This implies there is not an efficient algorithm for the NP Class.

Algorithms, Complexity, SAT, NP, Quantum Computation.
68Q10, 68Q12,68Q19,68Q25.

1 Introduction

My previous works over the NP class is [1], [2], and [3]. In the last one, the classical decision problem, the Boolean Satisfiability Problem, named SAT was used to state a lower bound for its complexity.

As a general framework, my technique consists: 1) to study general problem, 2) to determine a simple reduction, and 3) to analyze for trying to build an efficient algorithm for the simple problem. I like to explains that to build an algorithms to determine a complexity bound has more than I depicts in [3]. I take the approach by similarities from applied mathematics: the well-know optimization conditions, the search inside of a region or outside of it, and fixed point method. My article [3] focus on describing the fixed point and probabilistic approach. This article is a commentary study of the decision SAT, the changes in the algorithms presented here do not change my main result for NP's complexity but they clarifies details.

This paper focus in the the SAT's properties, objections and proofs about the algorithms for solving an extreme case problem of SAT (also I called reduced SAT or Simple SAT, see section 2.). Hereafter, SSAT states Simple SAT. The

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following section depicts SAT and SSAT and their properties. The next section depicts the algorithms for SSAT, and the complexity for an extreme SSAT is depicted in the next section. Some parts of [3] are repeated here to make this article self-content.

2 SAT and Simple SAT

A Boolean variable only takes the values: 0 (false) or 1 (true). The logical operators are **not**: $\overline{x}$; **and**: $\wedge$, and **or**: $\vee$.

Hereafter, $\Sigma = \{0, 1\}$ is the corresponding alphabet, x is a binary string in Σ^n means its corresponding number in $[0, 2^n - 1]$ and reciprocally. The inner or fixed point approach means to take the data from the translation of the problem's formulas, and outside or probabilistic approach means to take randomly the data from the problem's search space.

A SAT(n, m) problem consists to answer if a system of m Boolean formulas in conjunctive normal form over n Boolean variables has an assignation of logical values such the system of formulas are true.

The system of formulas is represented as a matrix, where each row correspond to a disjunctive formula. By example, let SAT$(4, 4)$ be

$$\begin{array}{lllll} & (x_3 & \vee \overline{x}_2 & & \vee x_0) \\ \wedge & & (x_2 & \vee x_1 & \vee x_0) \\ \wedge & & (\overline{x}_2 & \vee x_1 & \vee x_0) \\ \wedge & (x_3 & & & \vee \overline{x}_0). \end{array}$$

This problem is satisfactory. The assignation $x_0 = 1, x_1 = 0, x_2 = 1$, and $x_3 = 1$ is a solution, as it is depicting by substituting the Boolean values:

$$\left.\begin{array}{lllll} & (1 & \vee 0 & & \vee 1) \\ \wedge & & (1 & \vee 0 & \vee 1) \\ \wedge & & (0 & \vee 0 & \vee 1) \\ \wedge & (1 & & & \vee 0) \end{array}\right. \equiv 1.$$

It is important to note that the requirement of rows with the same number of Boolean variables in a given order is a simple reduction for studying SAT. This paper focuses in this simple formulation of SAT. SSAT(n, m) is a SAT where its m Boolean row formulas have the same length and the Boolean variables are in each row are in the same order, $x_{n-1}, \ldots, x_0$.

For any SSAT, each row of the system of Boolean formulas can be translated into a set of binary numbers.

Each row of SSAT maps to a binary string in Σ^n, with the convention: $\overline{x}_i$ to 0 (false), and x_i to 1 (true) in the i position. Hereafter, any binary string in Σ^n represents a binary number and reciprocally.

For example, given SAT$(2, 2)$:

$$\begin{array}{lll} & (\overline{x}_1 & \vee x_0) \\ \wedge & (x_1 & \vee x_0). \end{array}$$

It is traduced to:

$$\begin{array}{c}01\\11.\end{array}$$

The problem is to determine, does SSAT(n, m) have a solution? without previous knowledge.

3 Characteristics and properties of SSAT

Proposition 1. *1) A problem SAT can be transformed into an equivalent SSAT. 2) A problem SSAT is a SAT. 3) SSAT could be a subproblem of a problem SAT.*

Proof. 1) A SAT is transformed into an equivalent SSAT by algebraic procedures based in $F \equiv F \wedge (v \vee \overline{v})$, where F is a formula and v is a Boolean variable. 2) Any SSAT is a SAT with formulas of the same number of variables 3) On the other hand, SAT could have a subset of the Boolean formulas, that they can be arranged as a SSAT. □

For the cases 2 and 3) the complexity for solving SSAT is less than the complexity for solving SAT. The case 1) opens the possibility that for some SAT can be solved with less complexity than solving SSAT. By example, SAT$(2, 2)$ for x_1, x_0 under $(x_0) \wedge (\overline{x}_0) \equiv 0$ versus SSAT$(2, 4)$ $(\overline{x}_1 \vee \overline{x}_0) \wedge (\overline{x}_1 \vee x_0) \wedge (x_1 \vee \overline{x}_0) \wedge (x_1 \vee x_0) \equiv 0$. However, the first system can be see as the SSAT$(1, 2)$ $(x_0) \wedge (\overline{x}_0)$, which has no solution. This article focuses in study SSAT, in my next article, the complexity SSAT $\preceq$ SAT is depicted in detail.

Proposition 2.

1. *Any SAT(n, m) can be translated to a matrix of ternary numbers, and the ternary numbers are strings in $\{0, 1, 2\}^n$.*
2. *The search space of SSAT is less than the search space of SAT.*

Proof.
1. Taking the alphabet $\{0, 1, 2\}$. Each row of SAT is mapping to a ternary number, with the convention: $\overline{x}_i$ to 0 (false), x_i to 1 (true), and 2 when the variable x_i is no present.
2. By construction, $|\Sigma^n| = |\{0, 1\}^n| = 2^n \leq 3^n = |\{0, 1, 2\}^n|$.

□

The previous propositions justify to focus in SSAT. The former proposition states that sections of a SAT can be see as subproblem type SSAT. Moreover, it is sufficient to prove that there is not polynomial time algorithm for it.

By example, the previous SAT$(4, 4)$, it contains the following SSAT$(3, 2)$:

$$\begin{array}{rl} & (\ x_2 \ \vee x_1 \ \vee x_0) \\ \wedge & (\ \overline{x}_2 \ \vee x_1 \ \vee x_0). \end{array}$$

Proposition 3. *Given a binary number $b = b_{n-1}b_n \ldots b_0$. Then the Boolean disjunctive formula that correspond to the translation of $\overline{b}$ is 0.*

Proof. Without loss of generality, let x be $= x_{n-1} \vee \overline{x}_{n-2} \vee \ldots \vee \overline{x}_0$ the translation of b and $\overline{x} = \overline{x}_{n-1} \vee x_{n-2} \vee \ldots \vee x_0$ the translation of $\overline{b}$. Then $x \wedge \overline{x} = (x_{n-1} \vee \overline{x}_{n-2} \vee \ldots \vee \overline{x}_0) \wedge (\overline{x}_{n-1} \vee x_{n-2} \vee \ldots \vee x_0) = (x_{n-1} \wedge \overline{x}_{n-1}) \vee (\overline{x}_{n-2} \wedge x_{n-2}) \vee \ldots \vee (\overline{x}_0 \wedge x_0)$ $= 0$. □

The translation of the rows formulas of SSAT allows to define a table of binary numbers for SSAT. The matrix of binary values is an equivalent visual formulation of SSAT(n, m). The following boards have not a set of values in Σ to satisfy them:

x_1
1
0

x_2	x_1
0	0
1	1
0	1
1	0

I called unsatisfactory boards to the previous ones. It is clear that they have not a solution because each binary number has its binary complement. To find an unsatisfactory board is like order the number and its complement, by example: 000, 101, 110, 001, 010, 111, 011, and 100 correspond to the unsatisfactory board, i.e.:

000
111
001
110
010
101
011
100.

By inspection, it is possible to verify that the previous binary numbers correspond to a SSAT$(3, 8)$ with no solution because any binary number is blocked by its complement binary number (see prop. 3). By example, 000 and 111 correspond to $(x_2 \vee x_1 \vee x_0) \wedge (\overline{x}_2 \vee \overline{x}_1 \vee \overline{x}_0)$. Substituting by example $x_2 = 1, x_1 = 1, x_0 = 1$, we get $(1 \vee 1 \vee 1) \wedge (0 \vee 0 \vee 0) \equiv (1) \wedge (0) \equiv 0$.

Proposition 4. *SSAT(n, m) has different rows and $m < 2^n$. There is a satisfactory assignation that correspond to a binary string in Σ^n as a number from 0 to $2^n - 1$.*

Proof. Let s be any binary string that corresponds to a binary number from 0 to $2^n - 1$, where s has not its complement into the translated formulas of the given SSAT(n, m). Then s coincide with at least one binary digit of each binary number of the translated rows formulas, the corresponding Boolean variable is 1. Therefore, all rows are 1, i.e., s makes SSAT$(n, m) = 1$. □

The previous proposition point out when a solution $s \in [0, 2^n - 1]$ exists for SSAT. More important, SSAT can be see like the problem to look for a number s which its complements does not corresponded to the translated numbers of the SSAT's formulas.

Proposition 5. *$SSAT(n, 2^n)$'s rows correspond to the 0 to $2^n - 1$ binary numbers. Then it is an unsatisfactory board.*

Proof. The binary strings of the values from 0 to $2^n - 1$ are all possible assignation of values for the board. These strings correspond to all combinations of Σ^n, and by the prop. 3 SSAT$(n, 2^n)$ has not solution. □

This proposition 5 states that if $m = 2^n$ and SSAT has different rows, then there is not a solution. These are necessary conditions for any SSAT but these conditions a) different rows formulas and b) the number of rows formulas are previous knowledge.

As it is depicted below, it is possible to evaluate SSAT(n, m) as a logic circuit without substituting, and evaluating the Boolean formulas, i.e., without knowing the rows of SSAT.

Proposition 6. *Given $SAT(n, m)$. There is not solution, if L exists, where L is any subset of Boolean variables, with their rows formulas isomorphic to an unsatisfactory board.*

Proof. The subset L satisfies the proposition 5. Therefore, it is not possible to find satisfactory set of n values for SAT(n, m). □

Here, the last proposition depicts a necessary condition in order to determine the existence of the solution for SSAT. It is easy to understand but it is quite different to accept that SSAT has not solution, i.e., that SSAT is equivalent to an unsatisfactory board. The next propositions justifies focus in an extreme SSAT because solving some easy cases of SSAT can be solved in very efficient time without a satisfactory assignation as a witness.

Proposition 7. *Given $SSAT(n, m)$. If $m < 2^n$ then $SSAT(n, m)$ has a solution, such answer is found with complexity $\mathbf{O}(1)$.*

Proof. The rows of the given SSAT(n, m) do not correspond to all numbers in the search space $[0, 2^n - 1]$, even with repeated rows. Then, it exits a number which is not blocked.

The complexity is $\mathbf{O}(1)$, the only step corresponds to "if $m < 2^n$ then SSAT(n, m) has a solution". □

Proposition 8. *Given $SSAT(n, m)$. Let k be the number of failed candidates of the search space $[0, 2^n]$, such $k = k_1 + k_2$ where k_1 is the number of candidates that their translation is a formula of $SSAT(n, m)$, and k_2 is the number of candidates that their translation is a repeated formula in $SSAT(n, m)$.*

If $m - k_2 < 2^n$ then $SSAT(n, m)$ has a solution, such answer is found with complexity $\mathbf{O}(k)$.

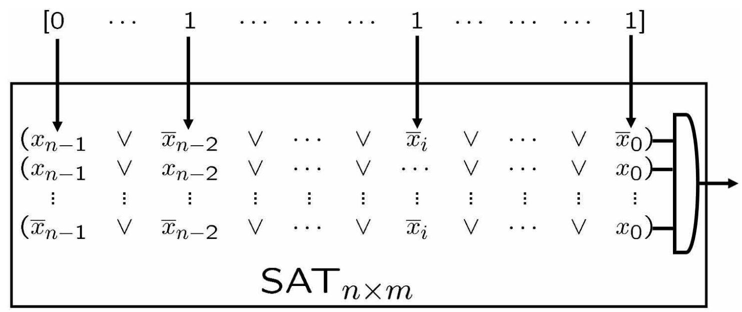

Figure 1: SAT(n, m) is a white of box containing a circuit of logical gates where each row has the same number of Boolean variables.

Proof. $k_1 + m - k$ is an estimation of the rows of the given SSAT(n, m). They do not correspond to all numbers in the search space $[0, 2^n - 1]$, because $k_1 + m - k$ $= k_1 + m - (k_1 + k_2) = m - k_2 < 2^n$. Then, SSAT$(n, m)$ has a satisfactory assignation, i.e., it is not a blocked board.

The complexity is $\mathbf{O}(k)$. It corresponds to the k tested candidates. □

The previous propositions do not estimate a witness x for verifying that SSAT$(n, m)(x) = 1$. SSAT(n, m) has a satisfactory assignation is implied by the fact that such SSAT(n, m) is not a blocked board.

SSAT can be see as a logic circuit, it only depends of the selection of the binary values assigned to n lines, each line inputs the corresponding binary value to its Boolean variable x_i. This is an important consideration because the complexity of the evaluation as a logic function is $\mathbf{O}(1)$. The figure 1 depicts SAT(n, m) as its logic circuit.

Proposition 9. *Given SSAT(n, m) as a circuit, and $M_{n\times m}$ the numbers of the translation of the SSAT(n, m)'s rows.*

1. *Let k be the translation of any row formula of SSAT(n, m).*
2. *Let k be any binary number, $k \in [0 : 2^n - 1]$.*

if SSAT$(n, m)(k) = 0$, then

1. *SSAT(n, m) $(\overline{k}) = 0$ and $k, \overline{k} \in M_{n\times m}$.*
2. *SSAT(n, m) $(k) = 0$ and $\overline{k} \in M_{n\times m}$.*

Proof. Without previous knowledge in the second case, the information that we have is SSAT(n, m) $(k) = 0$. It is caused by the translation of $\overline{k}$ in SSAT(n, m). On the former case when SSAT(n, m) $(k) = 0$ is not satisfied, it is because the complement of k blocks the system, i.e. SSAT(n, m) $(\overline{k}) = 0$ (see prop. 3) and $k, \overline{k} \in M_{n\times m}$. □

Proposition 10. $\Sigma = 0, 1$ *is an alphabet. Given SSAT*(n, m)*, the set* $\mathcal{S} = \{x \in \Sigma^n |$ *SSAT*$(n, m)(x)$*= 1* $\} \subset \Sigma^n$ *of the satisfactory assignations is a regular expression.*

Proof. $\mathcal{S} \subset \Sigma^n$. □

The last proposition depicts that a set of binary strings $\mathcal{S}$ of the satisfactory assignations can be computed by testing SSAT$(n, m)(x)$= 1, and the cost to determine $\mathcal{S}$ is 2^n, the number of different strings in Σ^n.

With $\mathcal{S} \neq \emptyset$ there is not opposition to accept that SSAT(n, m) has solution, no matters if m is huge and the formulas are in disorder or repeated. It is enough and sufficient to evaluate SSAT$(n, m)(x)$, $x \in \mathcal{S}$.

On the other hand, $\mathcal{S} = \emptyset$, there is not a direct verification. It is necessary, to validate how $\mathcal{S}$ is constructed. Solving SSAT(n, m)could be easy if we have the binary numbers that has not a complement in its translated rows. Also, because, $|\Sigma^n| = 2^n$ has exponential size, it could not be convenient to focus in the information of SSAT(n, m) with $m \gg 2^n$.

The complexity of the evaluation of SAT$(n, m)(y = y_{n-1}y_{n-2} \cdots\ y_1\ y_0)$ could be considered to be $\mathbf{O}(1)$. Instead of using a cycle, it is plausible to consider that SSAT$(n, m)$is a circuit of logical gates. This is depicted in figure 1. Hereafter, SAT(n, m) correspond to a logic circuit of "and", "or" gates, and the complexity of its evaluation is $\mathbf{O}(1)$.

Proposition 11.
SSAT(n, m) *has different row formulas, and* $m \leq 2^n$*. Any subset of* Σ^n *could be a solution for an appropriate SSAT*(n, m)*.*

Proof. $\emptyset$ is the solution of a blocked board., i.e., for any SSAT(n, m) with $m = 2^n$.

For $m = 2^n - 1$, it is possible to build a SSAT(n, m) with only x as the solution. The blocked numbers $[0, 2^n - 1] \setminus \{x, \overline{x}\}$ and x is translated and added to SSAT. By construction, SSAT$(n, m)(x) = 1$.

For f different solutions. Let S=$\{x_1, \ldots, x_f\}$ be the given solutions. Then the blocked numbers $B = \{y \in [0, 2^n - 1] \,|\, \forall x \in S_X, y \neq x$ or $\overline{y} \neq x\}$, where $S_X = \{s \in X \,|\, \overline{s} \notin X\}$. The numbers of S_X and $B \setminus X$ are translated and added to the resulting SSAT. □

It is prohibitive to analyze more than one iteration SSAT's formulas. For example, when $m \approx 2^n$, any strategy for looking solving SSAT could have m as a factor of the later iterations.

Proposition 12. $y \in \Sigma^n$ *and* $y = y_{n-1}y_{n-2} \cdots y_1y_0$*. The following strategies of resolution of SAT*(n, m) *are equivalent.*

1. *The evaluation of SAT*$(n, m)(y)$ *as logic circuit.*

2. *A matching procedure that consists verifying that each* y_i *match at least one digit* $s_i^k \in M_{n \times m}$*,* $\forall k = 1, \ldots, m$*.*

Proof. SAT$(n,m)(y)=1$, it means that at least one variable of each row is 1, i.e., each y_i, $i=1,\dots,n$ for at least one bit, this matches to 1 in s_j^k, $k=1,\dots,m$. □

The evaluation strategies are equivalent but the computational cost is not. The strategy 2 implies at least $m\cdot n$ iterations. This is a case for using each step of a cycle to analyze each variable in a row formulas or to count how many times a Boolean variable is used.

Proposition 13. *An equivalent formulation of SSAT(n,m) is to look for a binary number x^* from 0 to 2^n-1.*

1. *If $x^*\in M_{n\times m}$ and $\overline{x}^*\notin M_{n\times m}$ then SAT$(n,m)(x^*)=1$.*
2. *If $x^*\in M_{n\times m}$ and $\overline{x}^*\in M_{n\times m}$ then SAT$(n,m)(x^*)=0$. If $m<2^n-1$ then $\exists y^*\in[0,2^n-1]$ with $\overline{y}^*\notin M_{n\times m}$ and SAT$(n,m)(y^*)=1$.*
3. *if 2), then $\exists$ SAT$(n,m+1)$ such that 1) is fulfill.*

Proof.

1. When $x^*\in M_{n\times m}$ and $\overline{x}^*\notin M_{n\times m}$, this means that the corresponding formula of x^* is not blocked and for each Boolean formula of SAT$(n,m)(x^*)$ at least one Boolean variable coincides with one variable of x^*. Therefore SAT$(n,m)(x^*)=1$.
2. I have, $m<2^n-1$, then $\exists y^*\in[0,2^n-1]$ with $\overline{y}^*\notin M_{n\times m}$. Therefore, SSAT$(n,m)(y^*)=1$.
3. Adding the corresponding formula of y^* to SAT(n,m), a new SAT$(n,m+1)$ is obtained. By 1, the case is proved.

□

Proposition 14.
SSAT(n,m) has different row formulas, and $m\le 2^n$.
The complexity to solve SSAT(n,m) is $\mathbf{O}(1)$.

Proof. With the knowledge that $m<2^n$ the Boolean formulas of SSAT(n,m) does not correspond to a blocked board. It has not solution when $m=2^n$ and the SSAT(n,m)'s rows are different, i.e., it is a blocked board. □

This approach allows for verifying and getting a solution for any SSAT(n,m). By example, SAT$(6,4)$ corresponds to the set $M_{6\times4}$:

	$x_5=0$	$x_4=0$	$x_3=0$	$x_2=0$	$x_1=0$	$x_0=0$
	$\overline{x}_5\vee$	$\overline{x}_4\vee$	$\overline{x}_3\vee$	$\overline{x}_2\vee$	$\overline{x}_1\vee$	$\overline{x}_0)$
$\wedge($	$\overline{x}_5\vee$	$\overline{x}_4\vee$	$\overline{x}_3\vee$	$\overline{x}_2\vee$	$\overline{x}_1\vee$	$x_0)$
$\wedge($	$x_5\vee$	$x_4\vee$	$x_3\vee$	$x_2\vee$	$x_1\vee$	$\overline{x}_0)$
$\wedge($	$\overline{x}_5\vee$	$x_4\vee$	$x_3\vee$	$\overline{x}_2\vee$	$x_1\vee$	$x_0)$

x_5	x_4	x_3	x_2	x_1	x_0
0	0	0	0	0	0
0	0	0	0	0	1
1	1	1	1	1	0
0	1	1	0	1	1

.

The first table depicts that SAT$(6,4)(y = 000000) = 1$. The second table depicts the set $M_{6\times4}$ as an array of binary numbers. The assignation y corresponds to first row of $M_{6\times4}$. At least one digit of y coincides with each number of $\mathrm{M}_{n\times m}$, the Boolean formulas of SAT$(6,4)$. Finally, $y = 000000$ can be interpreted as the satisfied assignment $x_5 = 0$, $x_4 = 0$, $x_3 = 0$, $x_2 = 0$, $x_1 = 0$, and $x_0 = 0$.

SSAT(n,m)can be used as an array of m indexed Boolean formulas. In fact, the previous proposition gives an interpretation of the SSAT(n,m) as a type fixed point problem. For convenience, without exploring the formulas the SAT, my strategy is to look each formula, and to keep information in a Boolean array of the formulas of SAT by its binary number as an index for the array. At this point, the resolution SSAT(n,m) is equivalent to look for a binary number x such that SSAT$(n,m)(x) = 1$. The strategy is to use the binary number representation of the formulas of SSAT(n,m) in $\mathrm{M}_{n\times m}$.

SSAT as a function can be see as the function of a fixed point method, however, a satisfactory assignation could not belong to the binary translations of the SSAT's formulas. The advantage of taking the candidates from translations of SSAT's formulas is that for each failure, two numbers can be discarded (see prop. 9).

Furthermore, the equivalent between SSAT with the alternative formulation to determine if there is a binary string, which is not blocked in binary translations of the SSAT's formulas point out the lack of relationship between the rows of SSAT.

4 Extreme SSAT Problem

In section 2, the prop. 14 depicts that if SSAT's information includes that its rows are different then to answer is not a complex problem. In fact, SSAT's formulas are not necessary to review. The number of rows m and the fact that the SSAT's rows are different imply the answer without viewing inside the given problem SSAT.

Here, let us be critical, in order to build with precision an extreme problem. The extreme SSAT includes the parameters n (number of Boolean variables) and m the number of SSAT's rows. No information about the specific of SSAT's rows are given. But, the extreme problem could be a SSAT problem with only one binary string as solution or none, and it includes duplicate and disorder SSAT's rows. The selection of the unique solution is arbitrary, i.e., it could be any $s \in [0, 2^n - 1]$. Hereafter, $\mathcal{S} = \{x \in [0, 2^n - 1] \,|\, \mathrm{SSAT}(n,m)(x) = 1\}$.

The next propositions, depicts the difficult for determining a satisfactory assignation for an extreme SSAT.

Proposition 15. *Let n be large, and SSAT(n, m) an extreme problem, i.e., $|\mathcal{S}| \leq 1$, and $m \gg 2^n$.*

1. *The probability for selecting a solution ($\mathcal{P}_{ss}(f)$) after testing f different candidates ($f << 2^n$) is $\approx 1/2^{2n}$ (it is insignificant).*
2. *Given $C \subset [0, 2^n - 1]$ with a polinomial cardinality, i.e., $|C| = n^k$, with a constant $k > 0$. The probability that the solution belongs C ($\mathcal{P}_s(C)$) is insignificant, and more and more insignificant when n grows.*
3. *Solving SSAT(n, m) is not efficient.*

Proof.
Assuming that $|\mathcal{S}| = 1$.

1. The probability $\mathcal{P}_{ss}(f)$ corresponds to product of the probabilities for be selected and be the solution. For the inner approach (i.e., the f candidates are from the translations of the SSAT(n, m)'s rows) $\mathcal{P}_{ss}(f) = 1/\left(2^n - 2f\right)\cdot 1/2^n \approx 1/2^{2n} \approx 0$. For the outside approach (i.e., the f candidates are from the $[0, 2^n-1]$ the search space) $\mathcal{P}_{ss}(f) = 1/\left(2^n - f\right)\cdot 1/2^n \approx 1/2^{2n} \approx 0$.
2. $\mathcal{P}(C) = n^k/2^n$. Then $\mathcal{P}_s(C) = n^k/2^n \cdot 1/2^n$, and $lim_{n\to\infty} Kn^k/2^n$ (L'Hôpital's rule) $= 0^+$, $K > 0$. For n large, $2^n - Kn^k \approx 2^n$, and $Kn^k \ll 2^n$. Moreover, for the inner approach, $\mathcal{P}_{ss}(n^k) = 1/\left(2^n - 2n^k\right) \cdot 1/2^n \approx 1/2^{2n} \approx 0$. For the outside approach, $\mathcal{P}_{ss}(n^k) = 1/\left(2^n - n^k\right) \cdot 1/2^n \approx 1/2^{2n} \approx 0$.
3. In any approach, inner or outside, many rows of SSAT(n, m) have large probability to be blocked, because there is only one solution. Then the probability after f iterations remains $1/2^{2n} \approx 0$. It is almost impossible to find the solution with f small or a polinomial number of n.

Assuming that $|\mathcal{S}| = 0$. $\mathcal{P}_s = 0$.

1,2 For the inner approach and for the outside approach, $\mathcal{P}_{ss}(f) = 0$.

3 It is equivalent $\mathcal{S} = \emptyset \Leftrightarrow$ SSAT$(n, m)(x) = 0$, $\forall x \in [0, 2^n - 1]$. This means that it is necessary to test all the numbers in $[0, 2^n - 1]$.

□

One important similarity between the extreme SSAT as a numerical problem (see prop. 11) for one or none solution is the interpretation to guest such type of solution. It is like a lottery but with the possibility that there is not winner number. The exponential constant 2^n causes a rapidly decay as it depicted in fig. 2 where $t = 2^n - 1, 2^n - 8, 2^n - 32$.

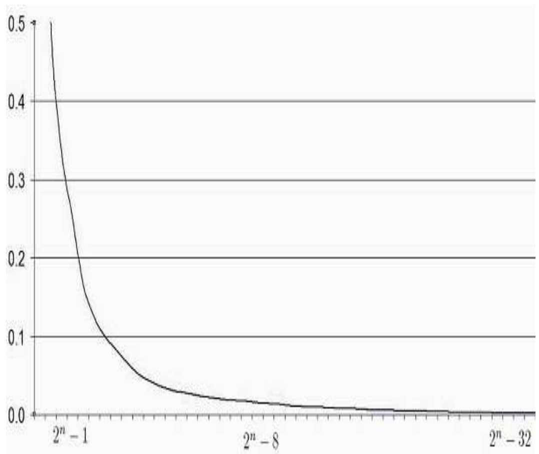

Figure 2: Behavior of the functions $P_e(t)$ and $P_i(t)$.

The interpretation of taking the extreme SSAT as a circuit for an electronic lottery behaves different when there is one winner number than when there is none. It is probably to wait for long time (it is an exponential waiting time) to get the winner number. People accept the winner ticket x^*, because a judge can show in an electronic board the result $\text{SSAT}(x^*) = 1$. It is unlikely to get the winner ticket in short time, but most of the people accept this case by testing the winner ticket. However, the case when there is not winner number is rejected, because the long time to wait to test all the numbers, and who can have the time, and be the unconditional and unbiased witness to testify that always the electronic board shows $\text{SSAT}(x) = 0$, $\forall x \in [0, 2^n - 1]$. Both cases are similar, and they point out that solving extreme SSAT takes an exponential time, no depending if a group of person does a lottery or a computer performs an algorithm.

Proposition 16. *There is not an efficient algorithm for solving extreme $SSAT(n, m)$.*

Proof. If such algorithm exists then it is capable for solving in polinomial time the equivalent number problem with one winner number or none in contradiction to the exponential time. □

5 Algorithms for SAT

The previous sections depict characteristics and properties of SSAT. The complexity for solving any $\text{SSAT}(n, m)$ needs at least one carefully review of the SSAT's rows, i.e., its complexity is related to the numbers m, and any algorithm for solving SSAT could have m has a factor related to its complexity. If it uses also the columns for substituting and simplifying by algebra the factor grows $m \cdot n$ at least. Also, the ordering and discarding repeated rows increased the complexity by $m\log_2(m)$. The properties depicted in section 2 indicate two source of data for solving SSAT(n, m), 1) its m rows or 2) the search space of all possible Boolean values for its variables (Σ^n). The second is large and m could be large also. Therefore, the efficient type of algorithms for solving SSAT must be doing in one way without cycles, and with the constraint that the total iterations must be related to $m < 2^n$, or 2^{n-1}, or 2^n. This is because the fixed point approach or inside search(taking candidates from the translation SSAT's formulas) and the outside approach or probabilistic approach (taking candidates from the search space $[0, 2^n - 1]$.

It is necessary to be sceptical and impartial, in order to accept the answer from a computers algorithm or a person. No matters if m is huge or SSAT is an extreme problem, without a proof or a clearly explication, I reject to accept such answer. This impose another characteristic for the algorithms for solving SSAT, they must provide a witness or something to corroborate that SSAT has been solved without objections.

A very simple algorithm to determine if SSAT has solution is in [3]. The algorithm is presented to solve SSAT by using the equivalent numerical formulation, more precisely for building an unsatisfactory board in a the table T.

Algorithm 1. ***Input:*** *SSAT*(n, m).

Output: *The answer if SSAT*(n, m) *has solution or not.* T *is an unsatisfactory board when SSAT*(n, m) *has not solution.*

Variables in memory: $T[0 : 2^n - 1] = -1$*: array of binary integer; address: integer; ct* $= 0$*: integer; k: binary integer.*

1. ***if*** $m < 2^n$ ***then***
2. ***output:*** *"SSAT*(n, m) *has a solution, its formulas do not cover* Σ^n*.";*
3. ***end if***
4. ***while not end(****SSAT*(n, m)***)***
5. $k = b_{n-1}b_{n-2} \ldots b_0 =$ ***Translate to binary formula*** *(SSAT*(n, m)*);*
6. ***if*** $k.[b_{n-1}]$ ***equal*** 0 ***then***
7. *address* $= 2 * k.[b_{n-2} \ldots b_0]$;
8. ***else***
9. *address* $= 2 * (2^{n-1} - k.[b_{n-2} \ldots b_0]) - 1$;
10. ***end if***
11. ***if*** *T[address]* ***equal*** -1 ***then***

12. $ct = ct + 1$;

13. $T[address] = k$;

14. ***end if***

15. ***if*** ct ***equal*** 2^n ***then***

16. ***output:*** *"There is not solution for SSAT*(n,m)*.*
    It has 2^n *different formulas.";*

17. ***stop***

18. ***end if***

19. ***end while***

20. ***output:*** *"SSAT*(n,m) *has a solution,*
    its formulas do not cover Σ^n*.";*

The previous algorithm is quite simple. It does not require to evaluate SSAT. The output has an equivalent formulation of the input SSAT, as a table of un unsatisfactory board, it writes "There is not solution for SSAT(n,m)". On the other hand, the algorithm writes "SSAT(n,m) has a solution", without any additional information, or witness.

It is reasonable to ask, do i accept the result of the previous algorithm?. The answers is "yes" but after carefully reviewing and verifying the correctness of the algorithm. If the answer of the algorithm is forgotten, it is possible to recall the answer from the table T, but it is not cheap. It is necessary to review in order to determine if there is a binary number without its complement or if all binary numbers are follow by its complement. In the former case, SSAT has a solution, in the second no. The objection is that the verification using T after running the algorithm is quite expensive.

Using the property of evaluating SSAT as circuit, the previous algorithm is modified to the next algorithm.

Algorithm 2. ***Input:*** *SSAT*(n,m).

Output: *An unsatisfactory board T when SSAT*(n,m) *has not a solution. A satisfactory assignation k when SSAT*(n,m) *has a solution.*

Variables in memory: $T[0:2^n-1]{=}{-1}$*: array of binary integer; address: integer;* $ct = 0$*: integer; k: binary integer.*

1. ***if*** $m < 2^n$ ***then***

2. ***output:*** *"SSAT*(n,m) *has a solution,*
    its formulas do not cover Σ^n*.";*

3. ***end if***

4. ***while not end(****SSAT*(n,m)***)***

```
    k = b_{n-1}b_{n-2}...b_0 = Translate to binary formula (SSAT(n,m));
    if SSAT(n,m)(k) equal 1 then
        output: "k is a solution for SSAT(n,m).";
        stop;
    end if;
    if k.[b_{n-1}] equal 1 then
        k = \overline{k};
    end if;
    address = 2 * k.[b_{n-2}...b_0];
    if T[address] equal -1 then
        T[address] = k;
        address = 2 * (2^{n-1} - \overline{k}.[b_{n-2}...b_0]) - 1;
        T[address] = \overline{k};
        ct = ct + 2;
    end if
    if ct equal 2^n then
        output: "There is not solution for SSAT(n,m).
    It has 2^n different formulas.";
        stop
    end if
end while
for k = 0 to 2^n - 1 do
    if T[k] equal -1 then
        output: "k is a solution of SSAT(n,m).";
        stop;
    end if;
for;
```

The previous algorithm solves the problem and it provides two type of witness: 1) an unsatisfactory board T when there is no solution, and 2) the satisfactory assignation k when there is a solution. It exploits the properties of SSAT(n, m) as a circuit, the inside search (i.e., the candidates come from the SSAT's formulas). Each failure eliminates two binary numbers, therefore the table T is building faster than the algorithm 1. The algorithm does not use a double linked list as the algorithms 2 and 3 in [3]. The drawback of this algorithm are the last lines. Here, the satisfactory assignation is founded but it is expensive with more the 2^n iterations. This could be changed by using a double linked list as in algorithms 2 and 3 in [3], this requires a lot of memory. The difference between them is that the former stopped with one satisfactory assignation and the second stopped after build $\mathcal{S}$.

The algorithms 3 and 4 in [3] are building using deterministic and probabilistic approach. They provides different type of witness to corroborate when SSAT has solution or not. The former gives a double linked list with the elements of $\mathcal{S}$ and the other gives a Boolean table T where the elements of $\mathcal{S}$ correspond to $i \in [0, 2^n - 1]$ such that $T[i] = 0$.

The situation for solving SSAT(n, m) is subtle. Its number of rows could be exponential, but for any SSAT$n, m)$, there are no more than 2^n different rows, then $m \gg 2^n$ means duplicate rows. It is possible to consider duplicate rows but this is not so important as to determine at least one solution in Σ^n. The search space Σ^n corresponds to a regular expression and it is easy to build by a finite deterministic automata (Kleene's Theorem) but in order. However, to test the binary numbers in order is not adequate. For m very large any source of binary number as candidates must be random and its construction be cheap. The next algorithm generates a random permutation the numbers from 0 to Mi.

Algorithm 3. ***Input:*** $T[0 : Mi] = [0 : Mi]$.
Output: $T[0 : Mi]$ *contains a permutation of the numbers from* 0 *to* Mi.
Variables in memory: $i = 0$*: integer; rdm, a=0 : integer;*

1. ***for*** *i:=0 to* $Mi - 1$
2.   ***if*** $T[i]$ ***equals*** i ***then***
3.   ***select uniform randomly*** $rdm \in [i + 1, Mi]$;
4. $a = T[rdm]$;
5. $T[rdm] = T[i]$;
6. $T[i] = a$;
7.   ***end if***
8. ***end for***
9. ***stop***

An important property of this algorithm is that it builds a permutation of the numbers 0 to Mi. None index coincide with the numbers in order.

Let floor() be a function, it returns the smallest integer less than or equal to a given number. Let rand() be a function that it returns a random real number in $(0, 1)$. The line **Select uniform randomly** $rdm \in [0, Mi - 1]$; could be implemented $k = \text{floor}(r \cdot Mi)$, where $r = rand()$, and $Mi > 0$, integer. Then $0 \leq k \leq Mi - 1$. In similar way, **Select uniform randomly** $rdm \in [i+1, Mi]$; could be implemented as $k = \text{floor}(r \cdot (Mi - i + 1.5)) + (i + 1)$.

The previous algorithm, is an alternative to change the line 4 in the probabilistic algorithm 4 in [3]:

4. **select uniform randomly** $k \in [0, 2^n - 1] \setminus \{i \,|\, T[i] = 1\}$;

Using the approach of the algorithm 3, the next algorithm solves SSAT(n, m) in straight forward using an outside approach. Here, each candidates is a random selection from $[0, 2^{n-1}]$.

Algorithm 4. ***Input:*** *n, SSAT*(n, m).

Output: *rdm, such that SSAT*$(n, m)(rdm) = 1$ *or SSAT has not solution.*

Variables in memory: $T[0 : 2^{n-1} - 1] = [0 : 2^{n-1} - 1]$*: integer;* $Mi = 2^{n-1} - 1$*: integer;* rdm, a*: integer.*

```
1.  if m < 2^n then
2.      output: "SSAT(n,m) has a solution,
        its formulas do not cover Σ^n.";
3.  end if
4.      if T[i] equals i then
5.  for i:=0 to Mi − 1
6.      if T[i] equals i then
        // select uniform randomly rdm ∈ [i + 1, Mi];
7.          rdm = floor(rand() · (Mi − i + 1.5)) + (i + 1);
8.          a = T[rdm];
9.          T[rdm] = T[i];
10.         T[i] = a;
11.     end if
12.     rdm = 0T[i];
13.     if SSAT(n,m)(rdm) equals 0 and
           SSAT(n,m)(¬rdm) equals 0 then
14.         continue
```

end if

if $SSAT(n,m)(rdm)$ **equals** *1* **then**

  output: *"rdm is a solution for* $SSAT(n,m)$*.";*

  stop;

else

  output: *"*$\overline{rdm}$ *is a solution for* $SSAT(n,m)$*.";*

  stop;

end if

end for

$rdm = 0T[Mi]$;

if $SSAT(n,m)(rdm)$ **equal** *1* **then**

  output: *"rdm is a solution for* $SSAT(n,m)$*.";*

  stop;

end if

if $SSAT(n,m)(\overline{rdm})$ **equal** *1* **then**

  output: *"*$\overline{rdm}$ *is a solution for* $SSAT(n,m)$*.";*

  stop;

end if

output: *"There is not solution for* $SSAT(n,m)$,
$SSAT(n,m)(x) = 0, \forall x \in [0, 2^{n-1}]$*.";*

stop;

The limit of the iterations to reach the answer is $Mi+1 = (2^{n-1}-1)+1 = 2^{n-1}$. Therefore, the complexity of the previous algorithm is $\mathbf{O}\left(Mi\right)$=$\mathbf{O}\left(2^{n-1}\right)$. No matters if the rows of SSAT(n,m) are duplicates or disordered or $m \gg 2^n$. The upper bound of the iterations is 2^{n-1} and the search space is $[0, 2^n-1]$ because a value $x \in [0, 2^{n-1}-1]$ is used to build $rdm = 0x \in [0, 2^n-1]$ and $\overline{rdm} \in [0, 2^n-1]$ are tested in the same iteration.

6 Complexity for SSAT

The prop. 11 depicts the complexity of solving SSAT and how to build a SSAT with some given set of solutions. By example, the following SSAT$(3,7)$ has one solution $x_2 = 0$, $x_1 = 1$, and $x_0 = 1$:

				Σ^3	$[0,7]$
	$\overline{x}_2\vee$	$\overline{x}_1\vee$	$\overline{x}_0)$	000	0
$\wedge($	$\overline{x}_2\vee$	$\overline{x}_1\vee$	$x_0)$	001	1
$\wedge($	$\overline{x}_2\vee$	$x_1\vee$	$\overline{x}_0)$	010	2
$\wedge($	$\overline{x}_2\vee$	$x_1\vee$	$x_0)$	011	3
$\wedge($	$x_2\vee$	$\overline{x}_1\vee$	$x_0)$	101	5
$\wedge($	$x_2\vee$	$x_1\vee$	$\overline{x}_0)$	110	6
$\wedge($	$x_2\vee$	$x_1\vee$	$x_0)$	111	7

By construction, the unique solution is the binary string of 3. It corresponds to the translation $(\overline{x}_2 \vee x_1 \vee x_0)$. It satisfies SSAT$(3,7)$, as the assignation $x_2 = 0$, $x_1 = 1$, and $x_0 = 1$. It is not blocked by 100, which corresponds to the missing formula $(x_2 \vee \overline{x}_1 \vee \overline{x}_0)$ (The complement of the formula 3). The other numbers $0,1,2$ are blocked by $5,6,7$.

Proposition 17. *Let SSAT$(n, 2^n - 1)$ be a problem with only one solution and its rows in ascendent order. Then the complexity by a binary search to determine the unique solution is* $\mathbf{O}(\log_2(2^n - 1)) \approx \mathbf{O}(n)$.

Proof. Without loss of generality the rows can be as the previous example SSAT$(3,7)$ in a table with indexes from $[0, 2^n - 2]$.

The following algorithm determines the unique solution:

Algorithm 5. ***Input:*** *SSAT(n,m) with only one solution and its rows in ascending order.*

Output: *The unique satisfactory assignation k.*

Variables in memory: $T[0 : 2^n - 2]$ = *(Translated SSAT'rows) : array of binary integer; l_i, r_i, m_i: integer.*

1. ***if*** $T[0]$ ***is not equals*** 0 ***then***
2.   ***output:*** *"0 is the solution.";*
3.   ***stop;***
4. ***end if***
5. ***if*** $T[2^n - 2]$ ***is equals*** $2^n - 1$ ***then***
6.   ***output:*** *"$2^n - 1$ is the solution.";*
7.   ***stop;***

end if

$l_i = 0$;

$r_i = 2^n - 2$;

while $((r_i - l_i) > 1)$ ***do***.

    $m_i = (l_i + r_i)/2$.

    if $T[m_i]$ ***is equals*** m_i ***then***

        $l_i = m_i$;

    otherwise

        $r_i = m_i$.

    end if

end while

output: *"$l_i + 1$ is the solution.";*

stop*;*

□

The previous proposition is based in the numerical translation of SSAT. The drawback of the previous binary search is that it only applies for solving special SSAT$(2, 2^n - 1)$ with different rows and in ascending order. When SSAT$(2, 2^n - 1)$'s rows are in disorder, the cost of sorting includes $\mathbf{O}(2^n - 1)$ by using the Address Calculation Sorting (R. Singleton, 1956) [4]. It has lineal complexity and is the less expensive sorting to my knowledge. In this case the complexity to determine the unique solution is $\mathbf{O}(2^n)$.

On the other hand, the no solution case has complexity $\mathbf{O}(1)$, knowing that SSAT$(n, 2^n)$ has different rows, there is nothing to look for. But again, to know that SSAT$(n, 2^n)$ has different rows, it has the cost of at least $\mathbf{O}(2^{n-1})$ by verifying at least one time the SSAT$(n, 2^n)$'s rows by using the algorithm 3 in [3].

The extreme SSAT problem is designed to test how difficult is to determine one or none solution without more knowledge than n the number of variables, and m the number of rows. It is extreme because $m \gg 2^n$ could be huge. This implies that SSAT'rows are repeated, and the inner approach is not convenient. It could take more than $m \gg 2^n$ iterations. Also, it does not help to know that SSAT could have one or none solution. As it is mentioned before, any algorithm must to solve SSAT without loops.

The algorithms 1,2, and 3 in [3] are based in the inner or fixed point approach, therefore solving the extreme SSAT could takes more than 2^n iterations ($m \gg 2^n$ is huge). They behave not stable for the extreme SSAT. The number of iterations is quite wide depending of $m \gg 2^n$. With many SSAT's rows repeated the inner approach or fixed point type method has not advantage using the elimination of two candidates for solving the extreme SSAT, it has to review the SSAT's rows but duplicates rows do not provide information for knowing is the solution or not solution is reached. It has the lower bound 2^{n-1} for special SSAT$(2^n, 2^n-1)$ because, it eliminates k and $\overline{k}$ when k cames from the translation of the SSAT's rows. But depending if the SSAT's rows are duplicates and disorder, it could behave quite different and makes a huge number of iterations ($\gg 2^n$) for an extreme SSAT. By example, if SSAT has the same row 2^n times at the beginning, after 2^n iterations the algorithm is far away for solving SSAT. This phenomena does not happen with the outside approach, after 2^n iterations the solution is reached.

The algorithm 4 is based in the outside approach. It uses a random search in $[0, 2^n - 1]$ by creating two candidates from $[0, 2^{n-1}]$. The candidates are $0x$ and $\overline{0x}$, $x \in [0, 2^{n-1}]$. The pay off is an stable behavior, no matters the extreme SSAT. Each candidate provides information that slowly and consistently, it reduces the distance to the solution. When there is no solution, this algorithm always takes 2^{n-1} iterations and it performs less than 2^{n-1} when there is one solution. The algorithm takes advantage of the evaluation of SSAT as a logic function in a circuit (see fig. 1)but it can not use the inner approachs property for eliminating two candidates in each failure test but it tests two candidates at same time.

The narrow behavior of the outside approach is the size of the search space $[0, 2^{n-1}]$. The wide behavior of the inner approach is caused when $m \gg 2^n$ and by the possibility for testing all SSAT's rows.

Proposition 18. *Let n be large, and let SSAT be an extreme problem, i.e., $|\mathcal{S}| \leq 1$. The algorithms 1, 2, and 3 in [3], and algorithms 1 (inner approach) behaves wide, and the algorithms 4 [3], and algorithm 4 (probabilistic and outside approach) behave narrow.*

Proof. The property depicted in prop. 3 relates k and its complement, it allows to eliminate two numbers when the candidate come from translation of a SSAT's formula. This is the inner approach or fixed point type method. For the extreme SSAT, any of the algorithms 1, 2, and 3 in [3], and algorithms 1 could iterates more than 2^n when the given SSAT's rows are repeated. In this case after 2^n iterations, it is possible to be far away of the solution. When there is not solution, the number of iterations could be around $m \gg 2^n$. It is a wide range of iterations from 1 to m with $m \gg 2^n$.

On the other hand, the algorithms 4 [3], and algorithm 4 (probabilistic and outside approach) uses SSAT as function and they explores the search space $[0, 2^n - 1]$ by creating two candidates from $[0, 2^{n-1}]$. It means that at most 2^{n-1} iterations are needed for solving any SSAT, even in the case of an extreme SSAT with $m \gg 2^n$.

□

$m \approx 2^n$		Existence			Construction		
SSAT(n, m)		Test: $m - r \leq 2^n$			$x \in$SSAT(n, m)		
rows	r duplicate rows	min	avg	max	min	avg	max
$m = 2$	0	1	1	1	1	2	3
$m < 2^n$	0	1	1	1	1	$m/2$	$m+1$
$m = 2^n - 1$	0	1	1	1	1	2^{n-1}	2^n
$m = 2^n$ (different rows)		1	1	1	1	1	1
2^n (unknow rows, 2^n different rows) SSAT(n, m) no solution							
$m = 2^n$	0	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
$m = 2^n + 1$	1	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
$m = 2^n + r$	r	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
2^n (unknow rows,$2^n - 1$ different rows) SSAT(n, m) unique solution							
$m = 2^n - 1 + 1$	1	2	2^{n-2}	2^{n-1}	1	2^{n-1}	2^{n-1}
$m = 2^n - 1 + r$	r	2	2^{n-2}	2^{n-1}	1	2^{n-1}	2^{n-1}

Table 1: Behavior of algorithm 2 for the extreme SSAT

Proposition 19. *Given an extreme SAT(n, m). It is not possible to verify in polynomial time the solution of it.*

Proof. This result follows from an extreme SSAT(n, m), $m \gg 2^n$. A sceptical person or a computer program must matched the huge data of SSAT(n, m) and the answer of the algorithms. He or it does not execute any of the algorithms, they just receive the results. When there is not solution, a table or an structure provide by the algorithm means that $\mathcal{S}$ is empty. All the algorithms here give an answer and a witness. It is simple to verify when there is a solution s^*, SSAT$(n, m)(s^*) = 1$. But, when the answer is no solution, he or it has an equivalent formulation of $\mathcal{S} = \emptyset$ or that the extreme SSAT(n, m) is equivalent to the special SSAT$(n, 2^n)$ with different rows. The corroboration can not consist in accepting the answer blindly $\mathcal{S} = \emptyset$ or that the extreme SSAT(n, m) is equivalent to the special SSAT$(n, 2^n)$. Also it is not sufficient testing some candidates with SSAT(n, m) but all. The corroboration of the equivalence between extreme SSAT(n, m) and special SSAT$(n, 2^n)$ needs at least 2^n iterations to match their rows. Without executing a complete and carefully checking and matching, the results of the algorithms themselves are not a corroboration that the original extreme SSAT(n, m) fulfill: SSAT$(n, m)(x) = 0$, $\forall x \in [0, 2^n - 1]$ when there is not solution. □

The tables 1 and 2 summarizes the complexity for solving the extreme SSAT. For solving extreme SSAT, the column existence depicts that the complexity is $\mathbf{O}(1)$ for almost all the cases but $m = 2^n$ with unknown rows. This is because there is not a property for implying $\forall x \in [0, 2^n - 1]$, SSAT$(n, m)(x) = 0$ but to verify that all SSAT's rows are different. For this case, the algorithms 2 and 4

$m \gg 2^n$		Existence			Construction		
SSAT(n, m)					$x \in [0, 2^{n-1}]$		
rows	r duplicate rows	min	avg	max	min	avg	max
2^n (unknow rows, 2^n different rows) SSAT(n, m) no solution							
$m = 2^n + r$	r	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
2^n (unknow rows,$2^n - 1$ different rows) SSAT(n, m) unique solution							
$m = 2^n - 1 + r$	r	1	2^{n-2}	2^{n-1}	1	2^{n-2}	2^{n-1}

Table 2: Behavior of algorithm 4 for the extreme SSAT

prove that there is not solution after testing all possible candidates.

For more details see the end of the section 4. This means that there is no a shortcut for verifying $\mathcal{S} = \emptyset$ for a given extreme SSAT(n, m).

Conclusions and future work

The results here does not change the SAT's complexity of the article [3]. It was interesting to analyze with more details that SSAT problems and algorithms behaves quite wide. Particularly, the inner or fixed point approach has not an advantage for eliminating two candidates for extreme SSAT(n, m) and it gives the wide behaviour. However, outside approach or probabilistic approach behaves stable with the upper bound 2^{n-1}.

The outside approach and the evaluation of SSAT as a circuit correspond to the probabilistic type of method allow to build the stable algorithm 4. This algorithm is a more detailed version of the probabilistic algorithm 4 of [3].

Moreover, for extreme SSAT (n, m) with $m \approx 2^n$ the complexity inside (alg.2) is similar to the outside (alg.4), i.e., $\mathbf{O}\left(2^{n-1}\right)$.

The main result is the impossibility to build an efficient algorithm for solving the decision SSAT, i.e., for knowing if it has a satisfactory assignation or not. The sceptical point of view needs proof to confirm or deny an answer. The algorithms in this paper always give some kind of witness or proof. When $m < 2^n$, there is a solution because the formulas of the given SSAT$(n, , m)$ do not cover the binary combination of the search space Σ^n. A satisfactory assignation when SSAT has solution is sufficient. But, a message when there is not solution do not substitute the detailed corroboration that SSAT(n, m) has 2^n different formulas or that $\forall x \in [0, 2^n - 1]$, SSAT$(n, m)(x) = 0$ with $m \gg 2^n$. The lack of an easy test to verify when there is not solution point out that there is not way for verifying a solution in polynomial time.

Extreme SSAT states that in order to solve it, at least one review of its search space (Σ^n) is necessary. This is done by splitting it into two spaces: $\mathbf{0}\Sigma^{n-1}$ and $\mathbf{1}\Sigma^{n-1}$ in at most 2^{n-1} iterations. Finally, this implies $\mathbf{O}(\text{SSAT}) = \mathbf{O}(2^{n-1}) \preceq$

$\mathbf{O}$(NP-Soft) $\preceq$ $\mathbf{O}$(NP-Hard).

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