A novel algorithm for solving the Decision Boolean Satisfiability Problem without algebra

Carlos Barrón-Romero *

April 27, 2016

Abstract

This paper depicts an algorithm for solving the Decision Boolean Satisfiability Problem using the binary numerical properties of a Special Decision Satisfiability Problem, parallel execution, object oriented, and short termination. The two operations: expansion and simplification are used to explains why using algebra grows the resolution steps. It is proved that its complexity has an upper bound of 2^{n-1} where n is the number of logical variables of the given problem.

Algorithms, Complexity, SAT, NP, CNF.
68Q10, 68Q12,68Q19,68Q25.

1 Introduction

This paper focuses in solving the classical Decision Boolean Satisfiability Problem (SAT) using the results in [3] for an special case of the Decision Boolean Satisfiability Problem, named Simple SAT (SSAT). In [2], the chapter 6 depicts Reducibility. This term means the ability to solve a problem by finding and solving simple subproblems. This idea is used here between SAT and SSAT.

I focused to the CNF version of SAT, which it is justified for the logical equivalence and the abroad literature. Talking about SAT is immediately related to NP Class of Problem and its algothms [9, 12, 13, 10, 11, 8, 6, 5, 7].

The dominant characteristic of SSAT is that all its formulas have the same number of logical variables. Briefly, the main results in [3] are the lower upper bound 2^{n-1} of the SSAT's algorithms for an extreme case with one or none solution, and such algorithms realize one lecture of the SSAT's or one exploration of the research space $[0, 2^n - 1]$ and they are numerical without algebra. Therefore the complexity for SSAT(n, m) (where n is the number of boolean variables and m the number of rows) is that the existence of the solution is $\mathbf{O}(1)$ when $m < 2^n$ and it is bounded by $\mathbf{O}2^{n-1})$ for $m \geq 2^n$ or $m \gg 2^n$. On the other hand, in [4]

*Universidad Autónoma Metropolitana, Unidad Azcapotzalco, Av. San Pablo No. 180, Col. Reynosa Tamaulipas, C.P. 02200. MEXICO.

it is depicted that for SSAT does not exist an efficient algorithm for solving it. This shortly, because the extreme problem SSAT(n,m) with one or none solution and repeated rows, $(m \gg 2^n)$ requieres one exploration of the rows. The extreme problem SSAT(n,m) can be builded selecting randomly one number or none in $[0, 2n-1]$. Then the only way to infers the solution is verifying that the SSAT(n,m)'s rows corresponds to $2^n - 1$ of 2^n different formulas. On the other hand, the extreme problem SSAT(n,m) corresponds to a very easy SAT(n,m'). This article analyzes how to solve SAT using properties of SSAT and without using algebra ([4], and [3]).

The importance of the SAT's complexity is that SAT is a complete NP problem, and if it has an efficient algorithm then any NP problem can be solved efficiently. On the other hand, it has not an efficient algorithm, then **O**(SAT) $\preceq$ **O**(NP-Soft) $\preceq$ **O**(NP-Hard). In [3], it is depicted that **O**(SSAT) $\preceq$ **O**(SAT) because SSAT could be see as a subproblem of SAT. Here, a carefully resume of the preeminent properties of the SSAT and SAT are depicted and used to build an special algorithm for solving any SAT.

Some propositions of my previous works over the NP class, SAT, and SSAT in [1], [2], [4], and [3] are repeated for making this article self-content. It is worth to mention that the binary numerical approach depicted here is for solving any SAT with one lecture of the SAT's rows with the upper bound 2^{n-1} steps.

Section 2 contains basic definition, concepts and conventions used. The section 3 depicts the propositions that they characterize necessary and sufficient conditions for building an algorithm for SAT. The next section 4 depicts the algorithms of the parallel algorithm for solving SAT. It is proved that lower upper bound of the iterations for solving SAT is 2^n, where n is the number of logical variables. Section 5 discusses and justifies why the strategies of using SSAT's properties is less complex for solving SAT than algebra procedures. Particularly, two operations, expansion and simplification are depicted to connect SSAT and SAT. The last section depicts my conclusions and future work.

2 Notation and conventions

A boolean variable only takes the values: 0 (false) or 1 (true). The logical operators are **not**: $\overline{x}$; **and**: $\wedge$, and **or**: $\vee$.

Hereafter, $\Sigma = \{0,1\}$ is the corresponding binary alphabet. A binary string , $x \in \Sigma^n$ is mapped to its corresponding binary number in $[0, 2^n - 1]$ and reciprocally. Moreover, $x_{n-1} \vee \overline{x}_{n-2} \ldots x_1 \vee x_0$ corresponds to the binary string $b_{n-1}b_{n-2}\ldots b_1 b_0$ where $b_i = \begin{cases} 0 & \text{if } \overline{x}_i, \\ 1 & \text{otherwise.} \end{cases}$

Such translation has no cost, if x and $\overline{x}$ are represented in the ASCII code by a (097) and â (226).

The table for the translation is C

index	C
$\vdots$	$\vdots$
097	1
$\vdots$	$\vdots$
226	0
$\vdots$	$\vdots$

The clause $(x_{n-1} \vee \overline{x}_{n-2} \ldots x_1 \vee x_0)$ is written as $\mathrm{a}_{n-1}\ \hat{\mathrm{a}}_{n-2}\ \ldots\ \mathrm{a}_1\ \mathrm{a}_0$.

The translation is $C[c_{n-1}]C[c_{n-2}]\ldots C[c_1]C[c_0]$ where c_i is the logical variable a_i or $\hat{\mathrm{a}}_i$ in the position i. Hereafter, the SAT's formulas are translated to its corresponding binary string.

A SAT(n, m) problem consists to answer if a system of m clauses or row boolean formulas in conjunctive normal form over n boolean variables has an assignation of logical values such the system of formulas are true.

The system of formulas is represented as a matrix, where each clause or row formula corresponds to a disjunctive clause. By example, let SAT$(4, 3)$ be

$$
\begin{array}{cccccc}
 & (x_3 & \vee\, \overline{x}_2 & & & \vee\, x_0) \\
\wedge & & (x_2 & \vee\, x_1 & & \vee\, x_0) \\
\wedge & & (\overline{x}_2 & \vee\, x_1 & & \vee\, x_0) \\
\wedge & (x_3 & & & & \vee\, \overline{x}_0).
\end{array}
$$

This problem is satisfactory. The assignation $x_0 = 1, x_1 = 0, x_2 = 1$, and $x_3 = 1$ is a solution, as it is depicting by substituting the boolean values:

$$
\left.
\begin{array}{ccccc}
 & (1 & \vee\, 0 & & \vee\, 1) \\
\wedge & & (1 & \vee\, 0 & \vee\, 1) \\
\wedge & & (0 & \vee\, 0 & \vee\, 1) \\
\wedge & (1 & & & \vee\, 0)
\end{array}
\right. \equiv 1.
$$

A simple of SAT [SSAT(n, m)] is a SAT(n, m) with the requirement that its rows have the same number of boolean variables in a given order. The variables' order for SAT or SSAT do not imply to rewrite the problem, it has a constant cost that can be assumed at the lecture of each formula.

Hereafter, the boolean variables of SAT are identified by x with subindexes from $[0, n-1]$, i.e., $x_{n-1}, x_{n-2}, \ldots, x_1, x_0$. For the set $X = \{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}$, as in [2], Prop. 4. an enumeration for address identification of any subset of logical variables is as follows:

Set of Boolean Variables		$\mathbb{N}$
$\{\}$	$\leftrightarrow$	0
$\{x_0\}$	$\leftrightarrow$	$1 = \binom{n}{0}$
$\{x_1\}$	$\leftrightarrow$	$2 = \binom{n}{0} + 1$
$\vdots$	$\vdots$	$\vdots$
$\{x_1, x_0\}$	$\leftrightarrow$	$l_0 = \sum_{k=0}^{1} \binom{n}{k}$
$\{x_2, x_0\}$	$\leftrightarrow$	$l_1 = \sum_{k=0}^{1} \binom{n}{k} + 1$
$\{x_2, x_1\}$	$\leftrightarrow$	$l_2 = \sum_{k=0}^{1} \binom{n}{k} + 2$
$\vdots$	$\vdots$	$\vdots$
$X = \{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}$	$\leftrightarrow$	$2^n - 1 = \sum_{k=0}^{n-1} \binom{n}{k}$

The function $IV : 2^X \to [0, 2^n - 1]$ gives an unique identification as a natural number for any subset of boolean variables of X. It could be possible to define an address polynomial for IV, but for the moment the next algorithm can help for building the previos correspondence.

Algorithm 1. ***Input:*** *$x = \{x_k, x_{k-1}, \ldots, x_1, x_0\}$: Set of logical variables.*
Output: *ix: integer; // the unique index in $[0, 2^n - 1]$ for the set x.*
Variables in memory*: base: integer; v, t: variable set treated as a number of base $\{n-1, n-2, \ldots, 1, 0\}$*

$ix = 0$;
$k = |x|$; *// where $|\cdot|$ is the cardinality function.*
if k ***equals*** 0 ***then***
    output*: "ix.";*
    stop*;*
end if
$base = 0$;
  for $j = 0, k-1$ ***do***
    $base = base + \binom{n}{j}$; *// $\binom{\cdot}{\cdot}$ binomial coefficient*
end do
build $v = v_k, \ldots, v_0$; *// the lower set of variables in order of size k*
While $(x < v)$ ***do****.*
    $t = v$;
    repeat
        $t = t + 1$;
        if *variables in t different and in descending order* ***then***
            exit
        end if
    until *false;*
    $v = t$;
    $ix = ix + 1$;.
end while
output*: "$base + ix$.";*

stop;

Giving any SAT(n, m) and r any clause of it, then r can be associated to an unique and appropriate SSAT$(IV(r))$ using the algorithm 1, where r represents the set of boolean variables in the clause r with the convention that they subindex are in descending order.

The cross-join operator $(\times\theta)$ corresponds with two operations: a cross product and the natural join or (θ) join. the *theta* join is like the relational data base natural join operation. It is used between two set of variables r and r' and the solutions of SSAT$(\cdot).S$ as follow:

SSAT$(IV(r))$ $\times\theta$ SSAT$(IV(r'))$ =

1. **if** $r \cap r' = \emptyset$ **then** SSAT$(IV(r)).S \times$ SSAT$(IV(r')).S$.
2. **if** $r \cap r' \neq \emptyset$ and there are common values between SSAT$(IV(r)).S$ and SSAT$(IV(r')).S$ for the variables in $r \cap r'$ **then** SSAT$(IV(r)).S\, \theta_{r \cap r'}\, SSAT(IV(r')).S$
3. **if** $r \cap r' \neq \emptyset$ and there are not a common values between SSAT$(IV(r)).S$ and SSAT$(IV(r')).S$ for the variables in $r \cap r'$ **then** $\emptyset$.

In the case 1 and 2, SSAT$(IV(r)$ and SSAT$(IV(r')$ are compatibles. In the case 3) they are incompatibles, i.e., there is not a satisfactory assignation for both.

An example of the case 1) is the following φ_1 =SAT$(4, 5)$

$$\begin{array}{lll} & (x_3 \vee \overline{x}_2) & \\ \wedge & (x_3 \vee x_2) & \\ \wedge & (\overline{x}_3 \vee x_2) & \\ \wedge & & (\overline{x}_1 \vee \overline{x}_0) \\ \wedge & & (x_1 \vee x_0) \end{array}.$$

A SSAT$(2, 3)$ is the three first clauses of φ_1, its solution is $\begin{bmatrix} x_3 & x_2 \\ 1 & 1 \end{bmatrix}$.

A SSAT$(2, 2)$ is the last two clauses of φ_1, its solutions are $\begin{bmatrix} x_1 & x_0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Then the solutions of SAT$(4, 5)$ are $\{(1, 1)\} \times \{(0, 1), (1, 0)\}$ =
$\begin{bmatrix} x_3 & x_2 & x_1 & x_0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

An example of the case 2) is the following φ_2 =SAT$(4, 5)$

$$\begin{array}{lll} & (x_3 \vee \overline{x}_2) & \\ \wedge & (\overline{x}_3 \vee \overline{x}_2) & \\ \wedge & (\overline{x}_3 \vee x_2) & \\ \wedge & \quad (x_2 \vee & \overline{x}_1 \vee x_0) \\ \wedge & \quad (x_2 \vee & x_1 \vee \overline{x}_0) \end{array}.$$

A SSAT$(2,3)$ is the three first clauses of φ_2, its solution is $\begin{bmatrix} x_3 & x_2 \\ 0 & 0 \end{bmatrix}$.

A SSAT$(3,2)$ is the last two clauses of φ_2, its solutions are $\begin{bmatrix} x_2 & x_1 & x_0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Then SAT$(4,7)$ has solution, because 0 is the common value for x_2. The solutions are $\begin{bmatrix} x_3 & x_2 & x_1 & x_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

An example of the case 3) is the following φ_3 =SAT$(4,7)$

$$\begin{array}{cll} & (x_3 \vee \overline{x}_2) & \\ \wedge & (\overline{x}_3 \vee \overline{x}_2) & \\ \wedge & (\overline{x}_3 \vee x_2) & \\ \wedge & \quad (x_2 \vee & \overline{x}_1 \vee \overline{x}_0) \\ \wedge & \quad (x_2 \vee & \overline{x}_1 \vee x_0) \\ \wedge & \quad (x_2 \vee & x_1 \vee \overline{x}_0) \\ \wedge & \quad (x_2 \vee & x_1 \vee x_0) \end{array}.$$

A SSAT$(2,3)$ is the three first clauses of φ_3, its solution is $\begin{bmatrix} x_3 & x_2 \\ 0 & 0 \end{bmatrix}$.

A SSAT$(3,4)$ is the last four clauses of φ_3, its solutions are $\begin{bmatrix} x_2 & x_1 & x_0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Then SAT$(4,7)$ has not solution because there is not a common value for the variable x_2.

3 Properties for solving SAT

The translation of the SSAT(n,m)'s formulas can be arranged as a matrix $M_{n\times m} = \begin{bmatrix} b_0 \\ \vdots \\ b_m \end{bmatrix}$ where each binary number b_i corresponds to a each SSAT(n,m)'s clause.

SSAT can be see as a logic circuit, it only depends of the selection of the binary values assigned to n lines, each line inputs the corresponding binary value to its boolean variable x_i. This is an important consideration because the complexity of the evaluation as a circuit of a logic function is $\mathbf{O}(k)$ with k a

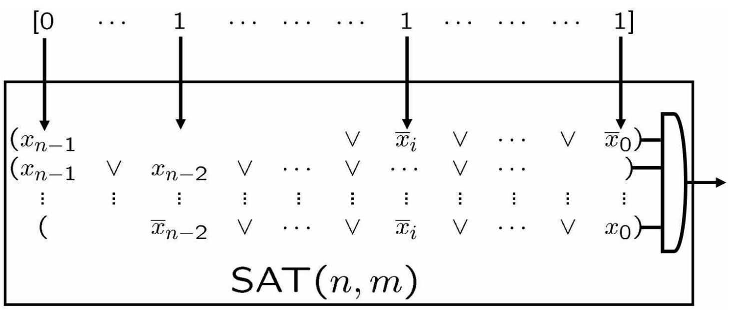

Figure 1: SAT(n, m) is a white of box containing a circuit of logical gates where each clause or row formula has the same number of boolean variables.

fixed time. k corresponds to the time that the electrons activate the circuit in parallel lines for each variable. Therefore, such evaluation can be considered as $\mathbf{O}(1)$. The figure 1 depicts SAT(n, m)'s logic circuit.

The inner approach means to take the candidates from the translation of the problem's formulas.

The outside approach means to the candidates from the problem's search space.

A deterministic approach means to take the candidates for solving a problem with the data in its given order.

A probabilistic approach means to take the candidates for solving a problem with no repetition and in random order, i.e., it likes a random permutation of the numbers $[0, M]$.

Proposition 1. *To any disjunctive clause $x = (x_{n-1}, \ldots, x_0)$ corresponds a binary number $b = b_{n-1}b_n \ldots b_0$. Then*

1. *$x \vee \overline{x} \equiv 0$ where $\overline{x}$ is the complement of x.*

2. *$b \vee \overline{b} \equiv 0$ where b and $\overline{b}$ correspond to the translation of x and $\overline{x}$.*

3. *$x(\overline{b}) \equiv 0$ and $\overline{x}(b) \equiv 0$ where the values of the boolean variables correspond to the bits of $\overline{b}$ and b.*

Proof. Without loss of generality, $x = x_{n-1} \vee \overline{x}_{n-2} \vee \ldots \vee \overline{x}_0$ the translation of b and $\overline{x} = \overline{x}_{n-1} \vee x_{n-2} \vee \ldots \vee x_0$ the translation of $\overline{b}$. Then $x \wedge \overline{x} = (x_{n-1} \vee \overline{x}_{n-2} \vee \ldots \vee \overline{x}_0) \wedge (\overline{x}_{n-1} \vee x_{n-2} \vee \ldots \vee x_0) \equiv (x_{n-1} \wedge \overline{x}_{n-1}) \vee (\overline{x}_{n-2} \wedge x_{n-2}) \vee \ldots \vee (\overline{x}_0 \wedge x_0)$ $\equiv 0$. □

The translation of the SSAT's rows allows to define a table of binary numbers. The following boards have not a satisfactory assignation in Σ and Σ^2:

x_1
1
0

x_2	x_1
0	0
1	1
0	1
1	0

I called unsatisfactory or blocked boards to the previous ones. It is clear that they have not a solution because each binary number has its binary complement. To find an unsatisfactory board is like to order the number and its complement, by example: 000, 101, 110, 001, 010, 111, 011, and 100 correspond to the unsatisfactory board:

000
111
001
110
010
101
011
100.

By inspection, it is possible to verify that the previous binary numbers correspond to SSAT$(3, 8)$ of the translated binary numbers with no solution because any binary number is blocked by its complement binary number (see prop. 1). By example, 000 and 111 correspond to $(x_2 \vee x_1 \vee x_0) \wedge (\overline{x}_2 \vee \overline{x}_1 \vee \overline{x}_0)$. Substituting by example $x_2 = 1, x_1 = 1, x_0 = 1$, we get $(1 \vee 1 \vee 1) \wedge (0 \vee 0 \vee 0)$ $\equiv (1) \wedge (0) \equiv 0$.

Proposition 2. *Let be SSAT(n, m) with m rows and $m < 2^n$. There is a satisfactory assignation that correspond to a binary string in Σ^n as a number from 0 to $2^n - 1$.*

Proof. Let s be any binary string that corresponds to a binary number from 0 to $2^n - 1$, where s has not its complement into the translated formulas of the given SSAT(n, m). Then s coincide with at least one binary digit of each binary number of the translated rows formulas, the corresponding logical variable is 1. Therefore, all rows are 1, i.e., s makes SSAT$(n, m) = 1$. □

The previous proposition point out when a solution $s \in [0, 2^n - 1]$ exists for SSAT. More important, SSAT can be see like the problem to look for a number s which its complements does not corresponded to the translated numbers of the SSAT's formulas.

Proposition 3. *Let be SSAT$(n, 2^n)$ where its rows correspond to the 0 to $2^n - 1$ binary numbers. Then it is an unsatisfactory board.*

Proof. The binary strings of the values from 0 to $2^n - 1$ are all possible assignation of values for the board. These strings correspond to all combinations of Σ^n, and by the prop. 1 SSAT$(n, 2^n)$ has not solution. □

This proposition 3 states that if $m = 2^n$ and SSAT has different rows, then there is not a solution.

Proposition 4. *For any SSAT(n, m) the conditions a) m rows formulas with $m < 2^n$ or b) m different formulas with $m = 2^n$ are sufficient conditions for deciding the solution of any SSAT(n, m) without further operations. Under any of these two conditions the complexity for determining the existence of the solution of SSAT is* $\mathbf{O}(1)$.

Proof. The results follow from the prop. 2 and from the prop. 3. □

Proposition 5. *Given SAT(n, m). There is not solution, if L exists, where L is any subset of boolean variables, with their rows formulas isomorphic to an unsatisfactory board.*

Proof. The subset L satisfies the proposition 3. Therefore, it is not possible to find satisfactory set of n values for SAT(n, m). □

Here, the last proposition depicts a condition to determine the no existence of the solution for SAT. This property is easy to implement in an algorithm using dynamic objects and when the set L is associated with SSAT$(IV(L))$. In order to determine that SAT contains a subproblem SSAT which it is an unsatisfactory board, it is the same to verify that SSAT$(IV(L))$ has not solution.

In similar way, the next proposition identifies when two SSAT can determine the no solution of SAT.

Proposition 6. *Given SAT(n, m). There is not solution, if L_1 and L_2 exist, where L_1 and L_2 are any subset of logical variables, such $L_1 \cap L_2 \neq \emptyset$, and the unique solution of SSAT$(IV(L_1))$ and the unique solution of SSAT$(IV(L_2))$ have no same values for the boolean variables in $L_1 \cap L_2$.*

Proof. The result follows because the unique solution of SSAT$(IV(L_1))$ blocks the solution of SSAT$(IV(L_2))$ and reciprocally. □

Proposition 7. *Given SSAT(n, m) as a circuit.*

1. *Let k be the translation of any clause of SSAT(n, m).*
2. *Let k be any binary number, $k \in [0 : 2^n - 1]$.*

if SSAT$(n, m)(k) = 0$, then

1. *SSAT$(n, m)(\overline{k}) = 0$ and the formulas of k and $\overline{k}$ are in SSAT(n, m).*
2. *SSAT(n, m) $(k) = 0$ and the translation of $\overline{k}$ is a formula of SSAT(n, m).*

Proof. 1. When SSAT$(n, m)(k) = 0$ is not satisfied, it is because the formula of $\overline{k}$ is 0. SSAT(n, m) contains the formulas of k and $(\overline{k}) = 0$ (see prop. 1).

2. $\mathrm{SSAT}(n,m)(k) = 0$, then by prop. 1 the translation of $\overline{k}$ is a formula of $\mathrm{SSAT}(n,m)$.

□

Proposition 8. *Let be $\Sigma = 0, 1$ an alphabet. Given SSAT(n,m), the set $\mathcal{S} = \{x \in \Sigma^n|$ SSAT$(n,m)(x)$= 1 $\} \subset \Sigma^n$ of the satisfactory assignations is a regular expression.*

Proof. $\mathcal{S} \subset \Sigma^n$. □

The last proposition depicts that a set of binary strings $\mathcal{S}$ of the satisfactory assignations can be computed by testing $\mathrm{SSAT}(n,m)(x)$= 1 for all $x \in [0, 2^n-1]$, and the cost to determine $\mathcal{S}$ is 2^n, the number of different strings in Σ^n.

With $\mathcal{S} \neq \emptyset$ there is not opposition to accept that $\mathrm{SSAT}(n,m)$ has solution, no matters if m is huge and the formulas are in disorder or repeated. It is enough and sufficient to evaluate $\mathrm{SSAT}(n,m)(x^*)$, $x^* \in \mathcal{S}$.

On the other hand, $\mathcal{S} = \emptyset$, there is not a direct verification. This implies the condition $\mathcal{S} = \emptyset \Leftrightarrow \forall x \in \Sigma^n$, $\mathrm{SSAT}(n,m)(x) = 0$. It is necessary, for verifying $\mathcal{S} = \emptyset$ to test all numbers in the search space.

Proposition 9.
SSAT(n,m) has different row formulas, and $m \leq 2^n$. Any subset of Σ^n could be a solution for an appropriate SSAT(n,m).

Proof. Any string of Σ^n corresponds to a number in $[0, 2^n - 1]$ and a SSAT's formula.

$\emptyset$ is the solution of a blocked board., i.e., for any $\mathrm{SSAT}(n,m)$ with $m = 2^n$.

For $m = 2^n - 1$, it is possible to build a $\mathrm{SSAT}(n,m)$ with only x as the solution. The blocked numbers $[0, 2^n - 1] \setminus \{x, \overline{x}\}$ and x are translated to SSAT's formulas. By construction, $\mathrm{SSAT}(n,m)(x) = 1$.

For f different solutions. Let $x_1, \ldots, x_f$ be the given expected solutions. Build the set C from the given solutions without any blocked pairs. Then the blocked numbers $[0, 2^n] \setminus \{y \in \Sigma^n | x \in C, y = x \text{ or } y = \overline{x}\}$ and the numbers of C are translated to SSAT's rows. □

Proposition 10. *Let be $y \in \Sigma^n$, $y = y_{n-1}y_{n-2}\cdots y_1 y_0$. The following strategies of resolution of SAT(n,m) are equivalent.*

1. *The evaluation of SAT$(n,m)(y)$ as logic circuit.*

2. *A matching procedure that consists verifying that each y_i match at least one digit $s_i^k \in M_{n\times m}$, $\forall k = 1, \ldots, m$.*

Proof. $\mathrm{SAT}(n,m)(y) = 1$, it means that at least one variable of each clause is 1, i.e., each y_i, $i = 1, \ldots, n$ for at least one bit, this matches to 1 in s_j^k, $k = 1, \ldots, m$. □

The evaluation strategies are equivalent but the computational cost is not. The strategy 2 implies at least $m \cdot n$ iterations. This is a case for using each step of a cycle to analyze each variable in a clause or to count how many times a boolean variable is used.

Proposition 11. *An equivalent formulation of SSAT(n,m) is to look for a binary number x^* from 0 to $2^n - 1$.*

1. *If $x^* \in \Sigma^n$ and $\overline{x}^* \notin M_{n\times m}$ then SAT$(n,m)(x^*) = 1$.*
2. *If $m \leq 2^n - 2$ and SSAT(n,m) has different rows then $\exists y^* \in [0, 2^n - 1]$ and SSAT$(n,m)(y^*) = 1$.*

Proof.

1. When $\overline{x}^* \notin M_{n\times m}$, this means that the corresponding formula of x^* is not blocked and for each SAT(n,m)'s clause at least one boolean variable coincides with one variable of x^*. Therefore SAT$(n,m)(x^*) = 1$.
2. $m \leq 2^n - 2$, then $\exists y_1, y_2 \in [0, 2^n - 1]$ with $y_1, y_2 \notin M_{n\times m}$. There are two cases. 1) $\overline{y}_2 = y1$, therefore, SSAT$(n,m)(y_1) = 1$. 2) $\overline{y}_1 \in M_{n\times m}$, therefore SSAT$(n,m)(\overline{y}_1) = 1$.

□

This previous proposition, depicts the equivalence between SSAT with the numerical problem to determine if there is a binary string, which is not blocked by the binary translations of the SSAT's formulas. Only k and its complement $\overline{k}$ are opposed (see prop. 1. This point out the lack of other type of relations between the rows of SSAT. More important, this proposition allows for verifying and getting a solution for any SSAT(n,m) without to evaluate SSAT(n,m) as a function. By example, SAT$(6,4)$ corresponds to the set $M_{6\times 4}$:

	$x_5 = 0$	$x_4 = 0$	$x_3 = 0$	$x_2 = 0$	$x_1 = 0$	$x_0 = 0$
	$\overline{x}_5\vee$	$\overline{x}_4\vee$	$\overline{x}_3\vee$	$\overline{x}_2\vee$	$\overline{x}_1\vee$	$\overline{x}_0)$
$\wedge($	$\overline{x}_5\vee$	$\overline{x}_4\vee$	$\overline{x}_3\vee$	$\overline{x}_2\vee$	$\overline{x}_1\vee$	$x_0)$
$\wedge($	$x_5\vee$	$x_4\vee$	$x_3\vee$	$x_2\vee$	$x_1\vee$	$\overline{x}_0)$
$\wedge($	$\overline{x}_5\vee$	$x_4\vee$	$x_3\vee$	$\overline{x}_2\vee$	$x_1\vee$	$x_0)$

x_5	x_4	x_3	x_2	x_1	x_0
0	0	0	0	0	0
0	0	0	0	0	1
1	1	1	1	1	0
0	1	1	0	1	1

.

How $y = 000000 \in \Sigma^6$ and $\overline{y} = 111111 \notin M_{6\times 4}$ then SSAT$(n,m)(000000) = 1$. On the other hand, $y_1 = 100100 \in \Sigma^6$ with $\overline{y}_1 = 011011 \in M_{6\times 4}$ then SSAT$(n,m)(011011) = 1$.

The next propositions, depicts the difficult for determining solving extreme SSAT.

Proposition 12. *Let n be large, and SSAT(n, m) an extreme problem, i.e., $|\mathcal{S}| \leq 1$, and $m \gg 2^n$.*

1. *The probability for selecting a solution ($\mathcal{P}_{ss}(f)$) after testing f different candidates ($f << 2^n$) is $\approx 1/2^{2n}$ (it is insignificant).*

2. *Given $C \subset [0, 2^n - 1]$ with a polynomial cardinality, i.e., $|C| = n^k$, with a constant $k > 0$. The probability that the solution belongs C ($\mathcal{P}_s(C)$) is insignificant, and more and more insignificant when n grows.*

3. *Solving SSAT(n, m) is not efficient.*

Proof.

Assuming that $|\mathcal{S}| = 1$.

1. The probability $\mathcal{P}_{ss}(f)$ corresponds to product of the probabilities for be selected and be the solution. For the inner approach (i.e., the f candidates are from the translations of the SSAT(n, m)'s rows) $\mathcal{P}_{ss}(f) = 1/\left(2^n - 2f\right) \cdot 1/2^n \approx 1/2^{2n} \approx 0$. For the outside approach (i.e., the f candidates are from the $[0, 2^n - 1]$ the search space) $\mathcal{P}_{ss}(f) = 1/\left(2^n - f\right) \cdot 1/2^n \approx 1/2^{2n} \approx 0$.

2. $\mathcal{P}(C) = n^k/2^n$. Then $\mathcal{P}_s(C) = n^k/2^n \cdot 1/2^n$, and $lim_{n\to\infty} Kn^k/2^n$ (L'Hôpital's rule) $= 0^+$, $K > 0$. For n large, $2^n - Kn^k \approx 2^n$, and $Kn^k \ll 2^n$. Moreover, for the inner approach, $\mathcal{P}_{ss}(n^k) = 1/\left(2^n - 2n^k\right) \cdot 1/2^n \approx 1/2^{2n} \approx 0$. For the outside approach, $\mathcal{P}_{ss}(n^k) = 1/\left(2^n - n^k\right) \cdot 1/2^n \approx 1/2^{2n} \approx 0$.

3. In any approach, inner or outside, many rows of SSAT(n, m) have large probability to be blocked, because there is only one solution. Then the probability after f iterations remains $1/2^{2n} \approx 0$. It is almost impossible to find the solution with f small or a polinomial number of n.

Assuming that $|\mathcal{S}| = 0$. $\mathcal{P}_s = 0$.

1,2 For the inner approach and for the outside approach, $\mathcal{P}_{ss}(f) = 0$.

3 It is equivalent $\mathcal{S} = \emptyset \Leftrightarrow$ SSAT$(n, m)(x) = 0$, $\forall x \in [0, 2^n - 1]$. This means that it is necessary to test all the numbers in $[0, 2^n - 1]$.

□

One important similarity between the extreme SSAT as a numerical problem (see prop. 9) for one or none solution is the interpretation to guest such type of solution. It is like a lottery but with the possibility that there is not winner number. The exponential constant 2^n causes a rapidly decay as it depicted in fig. 2 where $t = 2^n - 1, 2^n - 8, 2^n - 32$.

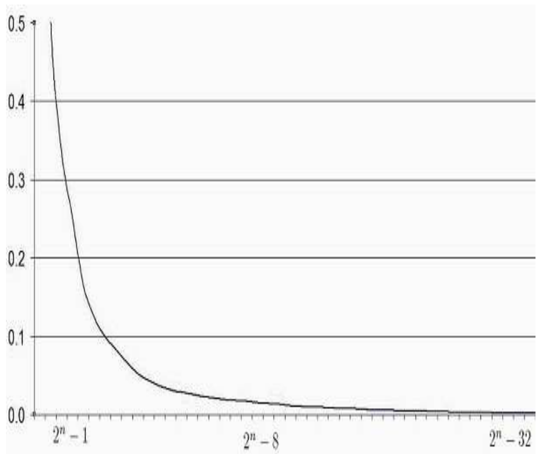

Figure 2: Behavior of the functions $P_e(t)$ and $P_i(t)$.

4 Algorithms for SAT

The previous sections depict characteristics and properties of SSAT.
The complexity for solving any SSAT(n, m) has two cases:

1. It is $\mathbf{O}(1)$ when $m < 2^n$.

2. It is $\mathbf{O}(m) \lessapprox \mathbf{O}(\mathbf{2}^{n-1})$, when giving $m \geq 2^n$. In this case, it is necessary at least one carefully review of the SSAT's rows versus the search space $[0, 2^n - 1]$.

The algorithms using the external approach are based in a random permutation with only one lecture of the SSAT's rows or they stop if a satisfiable assignation is found.

There are two source of data for solving SSAT(n, m), 1) its m rows or 2) the search space of all possible logical values for its variables (Σ^n). The second is large and m could be large also. Therefore, the efficient type of algorithms for solving SSAT must be doing in one way without cycles, and with the constraint that the total iterations must be related to $m < 2^n$, or 2^{n-1}, or 2^n. This is because the fixed point approach or inside search(taking candidates from the translation SSAT's formulas) and the outside approach or probabilistic approach (taking candidates from the search space $[0, 2^n - 1]$.

The situation for solving SSAT(n, m) is subtle. Its number of rows could be exponential, but for any SSAT(n, m), there are no more than 2^n different rows, then $m \gg 2^n$ means duplicate rows. It is possible to consider duplicate rows but this is not so important as to determine at least one solution in Σ^n. The search space Σ^n corresponds to a regular expression and it is easy to build by a finite deterministic automata (Kleene's Theorem) but in order. However, to test the binary numbers in order is not adequate.

Based in the algorithm 3 in [4], the update information for any SSAT($IV(r)$) is the following algorithm.

Algorithm 2. ***Input:*** *List of object:* $SSAT(n', m')$, $SSAT(IV(r))$, *rw: clause of* $SAT(n, m)$, *where* $r \subset X$.

Output: $SSAT(IV(r))$, T *: List of binary numbers such that,* $x \in T$, $SSAT(IV(r))(x) = 1$.

Variables in memory: $T[0 : 2^{n-1}]$*: list as an array of integers, double link structure previous, next : integer; ct:=0 : integer;* $first = 0$*: integer;* $last = 2^n - 1$*: integer;*

```
if SSAT(IV(r)) does not exist then
    create object SSAT(IV(r))
end if
with SSAT(IV(r))
    k = Translate to binary formula (rw);
    if T[k̄].previous not equal −1  or  T[k̄].next not equal −1 then
        // Update the links of T
        T[T[k̄].previous].next = T[k̄].next;
        T[T[k̄].next].previous = T[k̄].previous;
        if k̄ equal first then
            first := T[k̄].next;
        end if
        if k̄ equal last then
            last := T[k̄].previous;
        end if
        T[k̄].next = −1;
        T[k̄].previous = −1;
        ct := ct + 1;
    end if
    if ct equal 2^n then
        output: There is not solution for SAT_{n×m};
        everything stop;
    end if
end with
return
```

Algorithm 3. ***Input:*** n, $SAT(n, \cdot)$.

Output: *It determines if there is a solution or not of* $SAT(n, \cdot)$.

Variables in memory: *r: set of variables of X;* $SSAT(IV(r))$*: List of objects SSAT.*

```
While not EOF(SAT(n,·));
    r = SAT(n,·)'s clause;
    // Update the information of SSAT(IV(r))
    Algorithm 2(SSAT,r);
```

end while;
With the variables and the solutions of all $SSAT(IV(r))$ ***determine the set*** Θ ***of the corresponding*** $\times\theta$ *operation.*
if Θ ***is empty then***
    Output*: "Algorithm 3. There is not solution for the given SAT.*
        The solutions of all $SSAT(IV(r))$ *are incompatibles."*
    the process stops;
else
    output*: "Algorithm 3. There is solution for the given SAT.*
        Let be s any assignation $\in \Theta$. *It is a solution for* $SAT(n.m)$, *i.e.,*
        All $SSAT(IV(r))$ *are compatible."*
    the process stops;
end if

The next algorithm, is a version of the probabilistic algorithm 4 in [4]. It solves SAT(n,m) in straight forward using an outside approach, i.e, all the candidates are randomly and univocally selected from search space $[0, 2^n - 1]$.

Algorithm 4. ***Input:*** n, $SAT(n,\cdot)$.
Output: $s \in [0, 2^n - 1]$, *such that* $SAT(n,\cdot)(s) = 1$ *or SSAT has not solution (It determines if there is a solution or not of* $SAT(n,\cdot)$*).*
Variables in memory: $T[0 : 2^{n-1} - 1]$=$[0 : 2^n - 1]$*: integer;* Mi=$2^n - 1$*: integer;* rdm, a*: integer.*

---

for *i:=0 to* $2^{n-1} - 2$
    if $T[i]$ ***equals*** i ***then***
            // ***select randomly*** $rdm \in [i + 1, 2^{n-1} - 1]$*;*
        $rdm = floor(rand() \cdot (Mi - i + 1.5)) + (i + 1)$*;*
            // rand() return a random number in (0,1);
            // floor(x) return the lower integer of x
        $a = T[rdm]$*;*
        $T[rdm] = T[i]$*;*
        $T[i] = a$*;*
    end if
    $rdm = 0T[i]$*;*
    if $SAT(n,\cdot)(rdm)$ ***equals*** 1 ***then***
        output*: "Algorithm 4. There is solution for the given SAT.*
            The assignation x is a solution for $SAT(n,m)$*."*
        the process stops;
    end if
    if $SAT(n,m)(\overline{rdm})$ ***equals*** 1 ***then***
        output*: "Algorithm 4. There is solution for the given SAT.*
            The assignation $\overline{rdm}$ *is a solution for* $SAT(n,\cdot)$*."*
        the process stops;
    end if
end for
$rdm = 0T[2^{n-1} - 1]$;

if *$SAT(n,m)(rdm)$* ***equals*** 1 ***then***
  output*: "Algorithm 4. There is solution for the given SAT.*
    The assignation rdm is a solution for $SAT(n,\cdot)$."
  the process stops;
end if
if *$SAT(n,\cdot)(\overline{rdm})$* ***equals*** 1 ***then***
  output*: "Algorithm 4. There is solution for the given SAT.*
    The assignation $\overline{rdm}$ is a solution for $SAT(n,\cdot)$."
  the process stops;
end if
output*: "Algorithm 4. There is not solution for the given SAT.*
  Its rows cover all search space $[0,2^n]$, and they are blocked."
the process stops;

The complexity of the previous algorithm is $\mathbf{O}\left(2^{n-1}\right)$. No matters if the clauses of SAT$(n,\cdot)$ are huge or duplicates or disordered, i.e., $\gg 2^n$.

Algorithm 5. ***Input:*** *$\varphi =SAT(n,\cdot)$.*
Output*: The solution of $SAT(n,\cdot)$.*
Variables in memory*: List of object: $SSAT(n',m')$.*

---

run in parallel
  algorithm 3(n,φ);
  algorithm 4(n,φ);
end run

Proposition 13. *Let n be large, and $SAT(n,\cdot)$, $m \gg 2^n$. Then SAT is solved at most 2^{n-1} steps by running algorithm 5, which it runs in parallel the algorithms 3 and 4.*

Proof. Assuming enough time and memory. The two algorithms 3 and 4 run independently in parallel.

The algorithm 3 runs the algorithm 2 with the current r of SAT$(n,\cdot)$ until $m' = 2^{n'}$ or finishes at the end of the SAT(n,m)'s rows. Where $n' = IV(r).n$: number of boolean variables, and $m' = IV(r).m'$: number of different rows. When $m' = 2^{n'}$, the process stops because there is a blocked SSAT$(IV(r))$. After finish to read the rows, with the variables and the solutions of all SSAT$(IV(r))$, it determines from θ join operation the set Θ. if $\Theta = \emptyset$ then the process stops, all SSAT$(IV(r))$ are incompatibles. Otherwise, $\Theta \neq \emptyset$ and any $s \in \Theta$ is a solution for SAT$(n,\cdot)$.

On the other hand, the algorithm 4 takes two candidates at the same time $x = 0T[k]$ and $\overline{x}$). If one of them satisfies SAT$(n,\cdot)$ then stop. Otherwise, after all candidates are tested, SAT(n,m) has 2^n different rows then the process stops because SAT(n,m)'s rows cover all search space $[0,2^n]$, and they are blocked.

Finally, algorithm 4 limits the steps at most 2^{n-1} even if the number of clauses $\gg 2^n$. □

5 Complexity for SAT and SSAT

An extreme SSAT is a problem with one solution or none but without duplicates rows. For one solution, the simple comparison $m < 2^n$ allows to answer that the problem has a solution in one step. On the other hand, the no solution case has complexity $\mathbf{O}(1)$, knowing that SSAT$(n, 2^n)$ has different rows, there is nothing to look for. But again, to know that SSAT$(n, 2^n)$ has different rows, it has the cost of at least $\mathbf{O}(2^{n-1})$ by verifying at least one time the SSAT$(n, 2^n)$'s rows correspond to all combinations of Σ^n.

By example, the following SSAT$(3, 7)$ has one solution $x_2 = 0$, $x_1 = 1$, and $x_0 = 1$:

$$
\begin{array}{rlcc}
 & & \Sigma^3 & [0,7] \\
 & (\overline{x}_2 \vee \overline{x}_1 \vee \overline{x}_0) & 000 & 0 \\
\wedge & (\overline{x}_2 \vee \overline{x}_1 \vee x_0) & 001 & 1 \\
\wedge & (\overline{x}_2 \vee x_1 \vee \overline{x}_0) & 010 & 2 \\
\wedge & (\overline{x}_2 \vee x_1 \vee x_0) & 011 & 3 \\
\wedge & (x_2 \vee \overline{x}_1 \vee x_0) & 101 & 5 \\
\wedge & (x_2 \vee x_1 \vee \overline{x}_0) & 110 & 6 \\
\wedge & (x_2 \vee x_1 \vee x_0) & 111 & 7
\end{array}
$$

By construction, the unique solution is the binary string of 3. It corresponds to the translation $(\overline{x}_2 \vee x_1 \vee x_0)$. It satisfies SSAT$(3, 7)$, as the assignation $x_2 = 0$, $x_1 = 1$, and $x_0 = 1$. It is not blocked by 100, which corresponds to the missing formula $(x_2 \vee \overline{x}_1 \vee \overline{x}_0)$ (The complement of the formula 3). The other numbers $0, 1, 2$ are blocked by $5, 6, 7$.

An extreme SSAT has the next relation with a SAT:

1. The unique solution of SSAT$(n, 2^n - 1)$ corresponds to a SAT with n rows, where each row corresponds to each variable of the solution. By example, for the previous SSAT$(3, 7)$ its corresponding SAT(n, n) is $\begin{array}{l} (\overline{x}_2) \\ \wedge (x_1) \\ \wedge (x_0) \end{array}$.
2. The no solution case SSAT(n, m) corresponds to a blocked board, by example, SAT$(1, 2)$ could be $\begin{array}{l} (\overline{x}_0) \\ \wedge (x_0) \end{array}$.

The extreme SSAT problem is designed to test how difficult is to determine one or none solution knowing only n the number of variables, and m the number of rows. It is extreme because $m \gg 2^n$ could be huge. However, the corresponding versions of the extreme SSAT have two easy SAT problems. It is nor complicated but laborious to verify or build both SSAT and SAT. The next proposition allows to transform SAT in SSAT, and reciprocally.

Proposition 14. *Let F be a boolean formula and v a boolean variable, which is not in F. Then*

$$
(F) \quad \equiv \quad \begin{array}{rl} & (F \vee v) \\ \wedge & (F \vee \overline{v}) \end{array}
$$

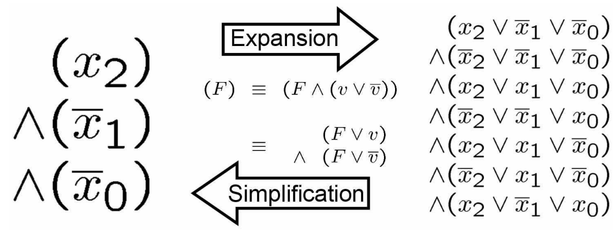

Figure 3: The relationship between SAT and SSAT.

Proof. The result follows from factorization and distribution laws:

$$(F) \quad \equiv \quad (F \wedge (v \vee \overline{v}))$$

. □

The previous proposition allows to define the operations expansion and simplification (see fig. 3):

1. Expansion. Add the two corresponding clauses for each boolean variable, which are not in F, where F is a factor or part of a boolean formula.

2. Simplification. Two clauses simplifies into one clause by the factorization: $(F \vee v) \wedge (F \vee \overline{v}) \equiv (F)$.

Proposition 15. *SSAT and SAT are equivalent, i.e., any SSAT can be transformed in SAT, and reciprocally by using by prop. 14.*

Proof. Giving a SAT, the formulas are completed by expansion (see fig. 3) from the previous proposition to build a SSAT. Reciprocally, Given a SSAT by using factorization it could be simplified to a SAT or there is nothing to do. In any case, SAT and its expansion SSAT or SSAT and its simplified SAT has the same set of solution by prop. 14. □

It is important to note, that for solving SSAT the complexity is bounded quasi lineal as a function of the number of SSAT's rows and it is bounded 2^n, because the size of the search space of all possible solutions Σ^n or $[0, 2^n - 1]$ but in function of the n the number of the logical variables of the problem. More important, it is trivial to solve SSAT when $m < 2^n$ and when $m = 2^n$ and SSAT$(n, 2^n)$'s rows are different.

On the other hand, when $m \gg 2^n$ it is by construction that at least one checking between SSAT and its search space is necessary to determine its solution. This up an objection to disqualify the extreme problem because it is by construction exponential in the number of SSAT's rows. Moreover, what could be a source of such type of problem or it is a theoretical curiosity. In my opinion, it is not a curiosity but a future technological issue. In [4], the algorithm

5 is an hybrid hardware-software over quantum computation and a SAT as an appropriate electronic logical circuit. The creation of novel electronic circuits is near to the level of the crystalline structures. This means that figure 1 could correspond to a crystalline structure. Here, for technical reasons existence and solutions are necessary to determine.

On the other hand, SAT as the general problem could have rows with any length and in any order. It is trivial to build a random SAT generator problems. The minimum parameter is n the number of logical variables, and the output are m the numbers of rows and the rows. Solving an arbitrary SAT by using algebra increase the complexity because it requires to compare and match rows and variables in some special order and with ad-hoc and appropriate structure for finding factor or parts where to apply the operations expansion or simplification (see fig. 3). This has as consequence more than one lecture or access of the SAT's rows, i.e., more than m operations, for ordering and matching SAT's rows and variables which it is not appropriate when m is large, i.e., $m \geq 2^n$ or $m \gg 2^n$.

The algorithm 5 and particularly the algorithm 3 do one lecture of the SAT's rows for extracting information of its subproblem SSAT. Together with the θ operation at the step 9, the algorithm 3 determines if such SSAT are compatibles or not. The iterations are less than m because the detection of a blocked SSAT is a sufficient condition to determine the no solution of the given SAT. On the other branch, the algorithm 4 divides the search space in two sections for testing two candidates at the same time in one iteration. It is possible to divide the search space in more sections but the candidates for testing in each step grows exponentially, 4,8, 2^k, ... The testing of the candidates can be in parallel and the complexity can be reduced to 2^{n-k} where 2^k are the candidates for testing. In our case, $2^1 = 2$, therefore the search space is exploring in 2^{n-1} steps. This means that it is necessary to have 2^k processors for testing 2^k candidates to get a lower upper bound of 2^{n-k} iterations. Taking in consideration that 2^k processor with $k \gg 0$ is not posible, the lower upper bound is 2^{n-1}.

The algorithm 3 is capable to process under enough memory and time with complexity $\mathbf{O}\left(|\varphi| + \text{time}(\times\theta\,\forall\,\text{SSAT}(IV(r))\right)$ any $\varphi = r, s$-SAT. The r, s-SAT formulation means formulas in CNF with clauses of r variables where any variable is repeated at most s times. In particular, any $r, 1$-SAT or $r, 2$-SAT or r, r-SAT can be solved as the algorithm of the state of art [9, 12, 13, 10]. Considering that $\text{time}(\times\theta\,\forall\,\text{SSAT}(IV(r))$ is solved by short-cut strategies. To detect $r, 1$-SAT is when $\bigcap_{r\in\varphi} IV(r) = \emptyset$, and the $\text{solutions}(\text{SSAT}(IV(r))) \neq \emptyset$. It is trivial and fast to create any $s \in \times$ $\text{solutions}(\text{SSAT}(IV(r)))$. It is similar for $r, 2$-SAT. Finally,

Proposition 16. *For any $\varphi = r, r$-SAT with exactly r clauses. The complexity to determine $\varphi \in$ SAT is* $\mathbf{O}(1)$.

Proof. By constructing, φ, it is a $\text{SSAT}(r, r)$, i.e., its parameters r, r are giving. The result follows by the proposition 4. □

Conclusions and future work

The results here confirm that there is a upper limit for the SAT's complexity (It was predicted in the article [4]). The outside approach and the evaluation of SSAT as a circuit correspond to the probabilistic type of method allow to build the stable algorithm 4. This algorithm is a more detailed version of the probabilistic algorithm 4 of [4]. It states the upper bound 2^{n-1} steps for solving any SAT.

The main result is the impossibility to build an efficient algorithm for solving SAT. This means, $\mathbf{O}\left(2^{n-1}\right)$=$\mathbf{O}$(SAT) $\preceq$ $\mathbf{O}$(NP Complete) $\preceq$ $\mathbf{O}$(NP-Soft) $\preceq$ $\mathbf{O}$(NP-Hard).

The consequences, when the result of the no existence of an efficient algorithm for the NP problems is accepted, are huge and paramount. Numerical intensive computers, supercomputers, high very large density of logical gates, quantum computation and the hybrid hardware-software will open a new era in the computer science.

References

[1] C. Barrón-Romero. Minimum search space and efficient methods for structural cluster optimization. *arXiv*, Math-ph:0504030-v4, 2005.

[2] C. Barrón-Romero. The Complexity of the NP-Class. *arXiv*, arxiv.org/abs/1006.2218, 2010.

[3] C. Barrón-Romero. Lower bound for the complexity of the boolean satisfiability problem. *ArXiv e-prints*, February, 2016.

[4] C. Barrón-Romero. Classical and Quantum Algorithms for the Boolean Satisfiability Problem. *ArXiv e-prints*, October, 2015.

[5] S. Cook. THE P VERSUS NP PROBLEM. *http://www.claymath.org/sites/default/files/ pvsnp.pdf*, 2000.

[6] L. Fortnow. The Status of the P Versus NP Problem. *Commun. ACM*, 52(9):78–86, Sept. 2009.

[7] D. Gutfreund, R. Shaltiel, and A. Ta-Shma. If NP Languages Are Hard on the Worst-Case, Then It is Easy to Find Their Hard Instances. *Comput. Complex.*, 16(4):412–441, Dec. 2007.

[8] G. J Woeginger. The P-versus-NP page. http://www.win.tue.nl/∼gwoegi/P-versus-NP.htm.

[9] P. Pudlák. *Mathematical Foundations of Computer Science 1998: 23rd International Symposium, MFCS'98 Brno, Czech Republic, August 24–28, 1998 Proceedings*, chapter Satisfiability — Algorithms and Logic, pages 129–141. Springer Berlin Heidelberg, Berlin, Heidelberg, 1998.

[10] C. A. Tovey. A simplified np-complete satisfiability problem. *Discrete Applied Mathematics*, 8(1):85 – 89, 1984.

[11] G. J. Woeginger. Exact algorithms for np-hard problems: A survey. *Combinatorial Optimization - Eureka, You Shrink!, LNCS*, pages 185–207, 2003.

[12] L. Zhang, C. F. Madigan, M. H. Moskewicz, and S. Malik. Efficient conflict driven learning in a boolean satisfiability solver. In *Proceedings of the 2001 IEEE/ACM International Conference on Computer-aided Design*, ICCAD '01, pages 279–285, Piscataway, NJ, USA, 2001. IEEE Press.

[13] L. Zhang and S. Malik. *Computer Aided Verification: 14th International Conference, CAV 2002 Copenhagen, Denmark, July 27–31, 2002 Proceedings*, chapter The Quest for Efficient Boolean Satisfiability Solvers, pages 17–36. Springer Berlin Heidelberg, Berlin, Heidelberg, 2002.