# Irreducible Quadrangulations of Surfaces

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Gloria Aguilar and Francisco Zaragoza

CINVESTAV (Mathematics) and UAM Azcapotzalco (Systems), Mexico

`gaguilar@math.cinvestav.mx` and `franz@correo.azc.uam.mx`

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# Surfaces and Euler genus

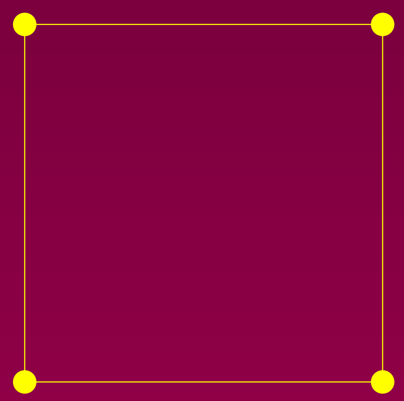
- Let $M_g$ be the sphere with $g$ handles attached.
- Let $N_g$ be the sphere with $g$ cross-caps attached.
- Their *Euler genuses* are $\gamma(M_g) = 2g$ and $\gamma(N_g) = g$.

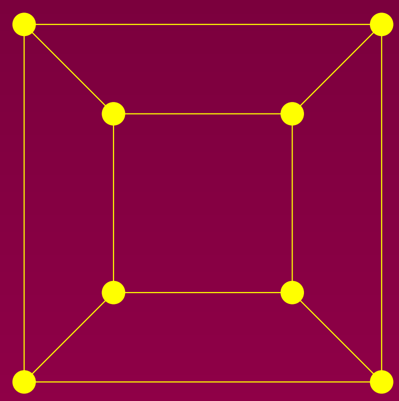
# Graphs and Euler genus

- Let $G$ be a simple graph.
- The *orientable genus* $\bar{\gamma}(G)$ of $G$ is the minimum $g$ such that $G$ is embeddable in $M_g$.
- The *non-orientable genus* $\tilde{\gamma}(G)$ of $G$ is the minimum $g$ such that $G$ is embeddable in $N_g$.
- The *Euler genus* $\gamma(G)$ of $G$ is $\min\{2\bar{\gamma}(G), \tilde{\gamma}(G)\}$.
- Note $\gamma(G) = \min\{\gamma(S) : G$ is embeddable in $S\}$.

# Quadrangulations

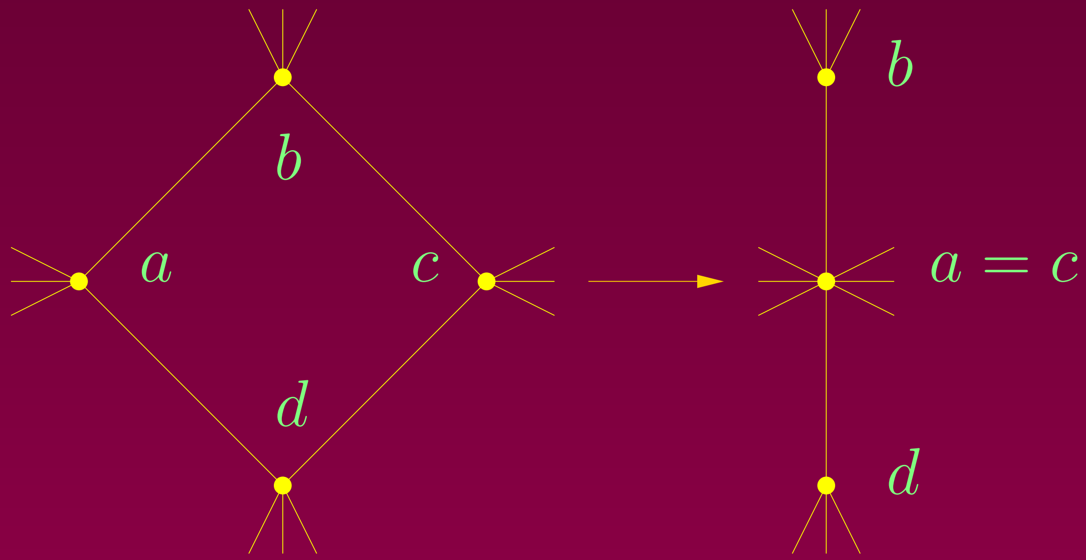
- A *quadrangulation* of a closed surface $S$ is a simple graph $G$ embedded on $S$ in such a way that all its faces have four (different) vertices.
- Two small quadrangulations of the sphere are the *square* and the *cube*:





# Irreducible Quadrangulations

- A face of a quadrangulation is *contractible* if this operation produces another quadrangulation:




- A quadrangulation is *irreducible* if it does not have *contractible* faces.

# Maximum Irreducible Quadrangulations

- Let $q(S)$ be the maximum number of vertices of an irreducible quadrangulation of a closed surface $S$.
- $q(S)$ is finite.
- $q(S) \leq 186\gamma(S) - 64$ (Nakamoto and Ota, 1995).

# Our Main Theorem

- **Main Theorem:** For every closed surface $S$ the following bound holds

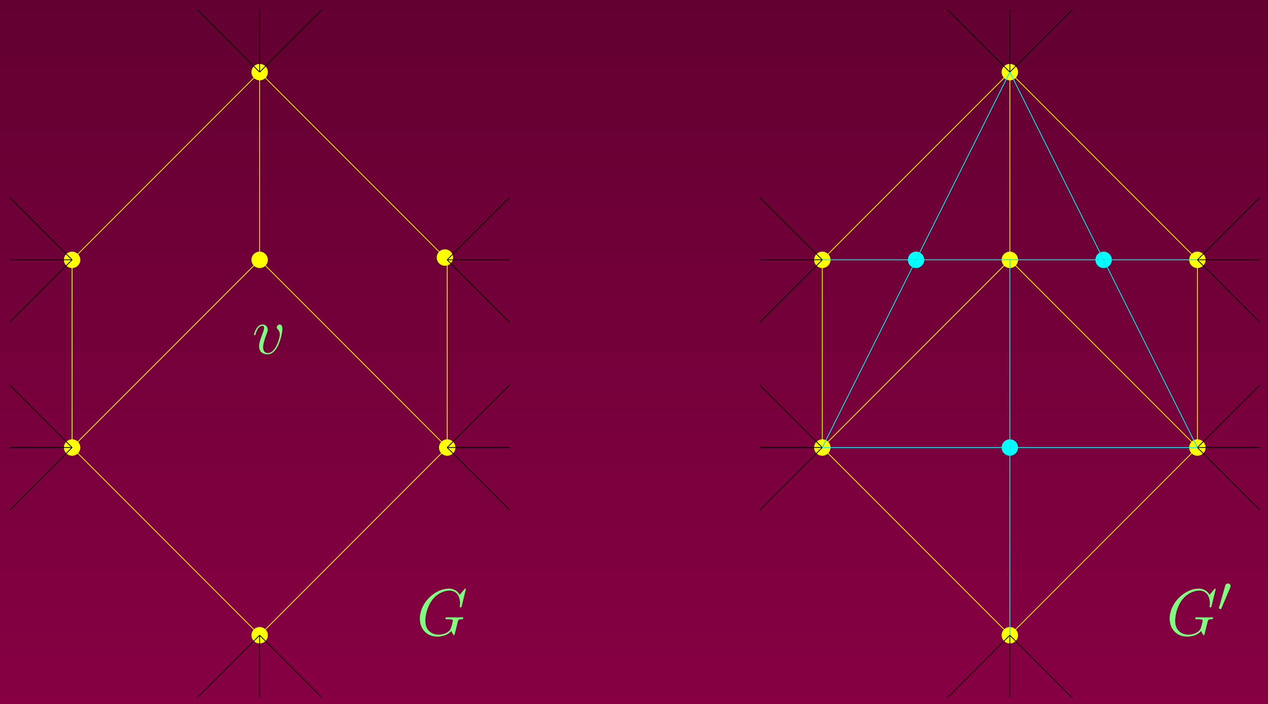
$$q(S) \leq 159.5\gamma(S) - 46.$$

- This is an improvement of about $16$ percent over the previous result.

# Some Useful Lemmas

1. Let $G_1, G_2$ and $G = G_1 \cup G_2$. If $|V(G_1) \cap V(G_2)| \leq 2$ then $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$ (Miller, 1987).
2. Let $G$ be an irreducible quadrangulation of $S \neq M_0$. $G$ has minimum degree $3$ (Nakamoto and Ota, 1995).
3. Let $G'$ be the graph embedded on $S$ obtained from $G$ by adding a vertex of degree $4$ to each of its faces. For $v \in V(G)$, let $H_v$ be the subgraph of $G'$ induced by $v$, the neighbours of $v$ in $G$, and the new vertices added to the neighbour faces of $v$. Then $\gamma(H_v) \geq 1$ (Nakamoto and Ota, 1995, for $\deg_G(v) \leq 4$).

# The graph $G'$



# Independent Set Lemma

- For $i \geq 3$ let $V_i$ be the set of vertices of degree $i$ in $G$.
- We say that $I \subseteq V(G)$ is *independent* if no two distinct vertices in $I$ are on the same face of $G$.
- **Independent Set Lemma:** Let $k \geq 3$. Then there exists an independent set $X \subseteq V_3 \cup V_4 \cup \cdots V_k$ such that

$$|X| \geq \sum_{i=3}^{k} \frac{|V_i|}{2i+1}.$$

- Nakamoto and Ota proved this for $k = 4$.

# Sketch of the Theorem (1)

- Let $k \geq 3$, let $X$ be as in the Independent Set Lemma, and $Y \subseteq V(G) \setminus X$ be the set of vertices on the same face of $G$ as a vertex in $X$.
- Construct a bipartite subgraph $B$ from $X$ and $Y$ which can be seen to be embeddable on $S$.
- Find a maximal subset $X' := \{v_1, v_2, \ldots, v_r\} \subseteq X$ so that you can apply Miller's Lemma to the sequence of graphs $H_{v_1}$, $H_{v_1} \cup H_{v_2}$, ..., $H_{v_1} \cup H_{v_2} \cup \cdots \cup H_{v_r}$ and obtain the bound $\gamma(S) \geq |X'|$.

# Sketch of the Theorem (2)

- Construct another bipartite subgraph $B'$ from $X'$ and $Y'$ which can also be embedded on $S$.
- Use the maximality of $X'$ to prove that $B'$ has at least $3(|X| - |X'|) + |Y'|$ edges.
- Use Euler's formula on $B'$ and properties of $X$ to prove that $4 - 2\gamma(S) \leq (2k+3)|X'| - |X|$.
- Use Euler's formula $|V_G| - |E_G| + |F_G| = 2 - \gamma(S)$ and elementary graph theory on $G$ to prove that

$$\sum_{i=3}^{k} (k-i+1)|V_i| \geq (k-3)|V(G)| - 4\gamma(S) + 8.$$

# Sketch of the Theorem (3)

- Let $n_k = \max_{3 \le i \le k}(2i+1)(k-i+1)$. Note that $n_3 = 7$, $n_4 = 14$, and $n_k = \frac{1}{2}(k+1)(k+2)$ for $k \ge 5$.
- Use the lower bound for $|X|$ in the Independent Set Lemma to obtain the inequality

$$4 - 2\gamma(S) \le -\frac{(k-3)|V(G)| + 4\gamma(S) - 8}{n_k} + (2k+3)|X'|.$$

- Or equivalently

$$\gamma(S) \ge \frac{(k-3)|V(G)| + 8 + n_k(4 - (2k+3)|X'|)}{2n_k + 4}.$$

# Sketch of the Theorem (4)

- We obtain two bounds for $\gamma(S)$ depending on $|X'|$. Eliminating this dependency we obtain

$$\frac{4+(5+2k)n_k}{k-3}\gamma(S) - \frac{4n_k+8}{k-3} \geq |V(G)|.$$

- For $k = 4$ this is $|V(G)| \leq 186\gamma(S) - 64$, the bound obtained by Nakamoto and Ota.
- For $k = 5$ this is $|V(G)| \leq 159.5\gamma(S) - 46$.
- For $k \geq 6$, the bound obtained for $|V(G)|$ is not better.

# Conclusions

- We improved the bound on $q(S)$.
- We have an alternate definition of *quadrangulation* for which the same bound is true and we are studying the corresponding irreducible quadrangulations of surfaces with low genus.
- We have also obtained improved bounds for the maximum size of *irreducible triangulations*.
- The old bound was $|V(G)| \leq 171\gamma(S) - 72$ and our new bound is $|V(G)| \leq 106.5\gamma(S) - 33$. This is an improvement of about $60$ percent.