Irreducible Triangulations of Surfaces

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Contents

Surfaces and Euler genus

- Let M_g be the sphere with g handles attached.
- Let N_g be the sphere with g cross-caps attached.
- Their *Euler genuses* are $\gamma(M_g) = 2g$ and $\gamma(N_g) = g$.

Graphs and Euler genus

- Let G be a simple graph.
- The *orientable genus* $\bar{\gamma}(G)$ of G is the minimum g such that G is embeddable in M_g.
- The *non-orientable genus* $\tilde{\gamma}(G)$ of G is the minimum g such that G is embeddable in N_g.
- The *Euler genus* $\gamma(G)$ of G is $\min\{2\bar{\gamma}(G), \tilde{\gamma}(G)\}$.
- Note $\gamma(G) = \min\{\gamma(S) : G$ is embeddable in $S\}$.

Triangulations

- A *triangulation* of a closed surface S is a simple graph G embedded on S so that each face is a triangle and any two faces share at most two vertices.
- K_4 is a triangulation of the sphere, K_3 is not.

Irreducible Triangulations

- A triangulation is *irreducible* if it has no *contractible* edges.
- All irreducible triangulations are known for $M_0, M_1, M_2, N_1, N_2, N_3, N_4$.

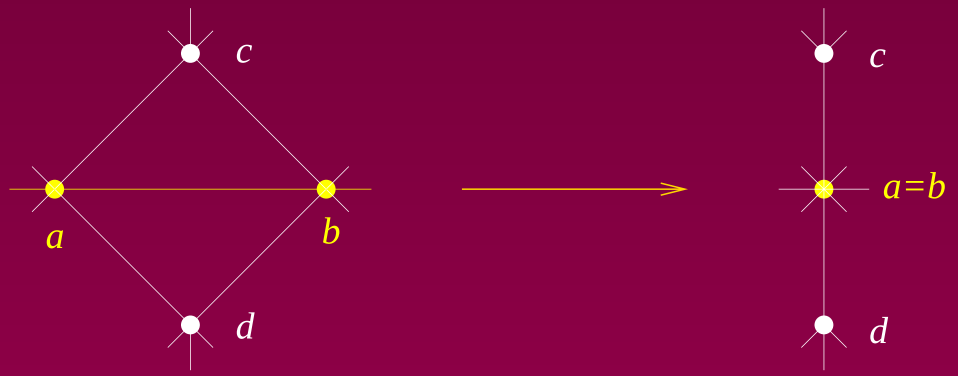

Figure 1: Contracting edge ab

Maximum Irreducible Triangulations

- Let $N(S)$ be the maximum number of vertices of an irreducible triangulation of a closed surface S.
- This number is finite (Barnette and Edelson, 1989).
- First bounds were $O(\gamma(S)^4)$ and $O(\gamma(S)^2)$.
- $N(S) \leq 171\gamma(S) - 72$ (Nakamoto and Ota, 1995).
- $N(S) \leq 120\gamma(S)$ for orientable S (Cheng, Dey, Poon, 2002).

Main Theorem and Main Lemma

- **Main Theorem:** $N(S) \leq 106.5\gamma(S) - 33$.
- **Main Lemma:** Let V_i be the set of vertices of degree $i \geq 4$ and let $k \geq 4$. There exists an independet set $X \subseteq V_4 \cup \cdots V_k$ such that

$$|X| \geq \sum_{i=4}^{k} \frac{|V_i|}{i+1}.$$

Sketch of Main Theorem (1)

- Choose an irreducible triangulation G of a surface S and apply Euler's formula.
- Choose $k \geq 4$, let X be as in main lemma and let $X' \subseteq X$ be maximal with a *nice* intersection property.
- Define two bipartite subgraphs of G with vertex sets $X \cup Y$ and $X \cup Y'$.
- The former gives $\gamma(S) \geq |X'|$ (Miller's lemma).
- The latter gives $4 - 2\gamma(S) \geq -|X| + (k+3)|X'|$.

Sketch of Main Theorem (2)

- Let n_k be the maximum value of $(i+1)(k-i+1)$ over $4 \le i \le k$.
- The main lemma and the last inequality give

$$\gamma(S) \ge \frac{(k-5)n + 12 + 4n_k - (k+3)n_k|X'|}{6+2n_k}.$$

- We have two lower bounds for $\gamma(S)$ (best bound depends on $|X'|$). Independently of $|X'|$ we obtain

$$\gamma(S) \ge \frac{(k-5)n + 4n_k + 12}{(k+5)n_k + 6}.$$

Sketch of Main Theorem (3)

- The lower bound is maximized for $k = 9$, that is

$$\gamma(S) \geq \frac{(9-5)n + 4n_9 + 12}{(9+5)n_9 + 6}.$$

- Since $n_9 = 30$ we obtain equivalently

$$n \leq 106.5\gamma(S) - 33.$$

Lower bounds for $N(S)$

- Nakamoto and Ota constructed irreducible triangulations to show that
  - $N(M_g) \geq 8g + 2$ for all $g \geq 1$.
  - $N(N_g) \geq 5g + 2$ for all $g \geq 1$.
- Sulanke (2006) improved these results to
  - $N(M_g) \geq \lfloor 17g/2 \rfloor$ for all $g \geq 1$.
  - $N(N_g) \geq \lfloor 11g/2 \rfloor$ for all $g \geq 1$.

Conclusions and Further Work

- All known bounds on $N(S)$ have been improved (either by us or by Sulanke).
- We have obtained improved bounds for *irreducible quadrangulations*.
- We have an alternate definition of *quadrangulation* and are studying irreducible quadrangulations of surfaces with low genus.
- We study *pentagonizations*, *hexagonizations*, etc.