Complexity of the Mixed Postman Problem with Restrictions on the Edges

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Contents

Mixed Graphs and Postman Tours

- A *mixed graph* $M = (V, E, A)$ consists of sets V of *vertices*, E of *edges*, and A of *arcs*.
- A *postman tour* of M is a closed walk W that traverses all vertices, edges, and arcs of M.
- An *arcs postman tour* of M traverses each edge of M exactly once.

An Arcs Postman Tour

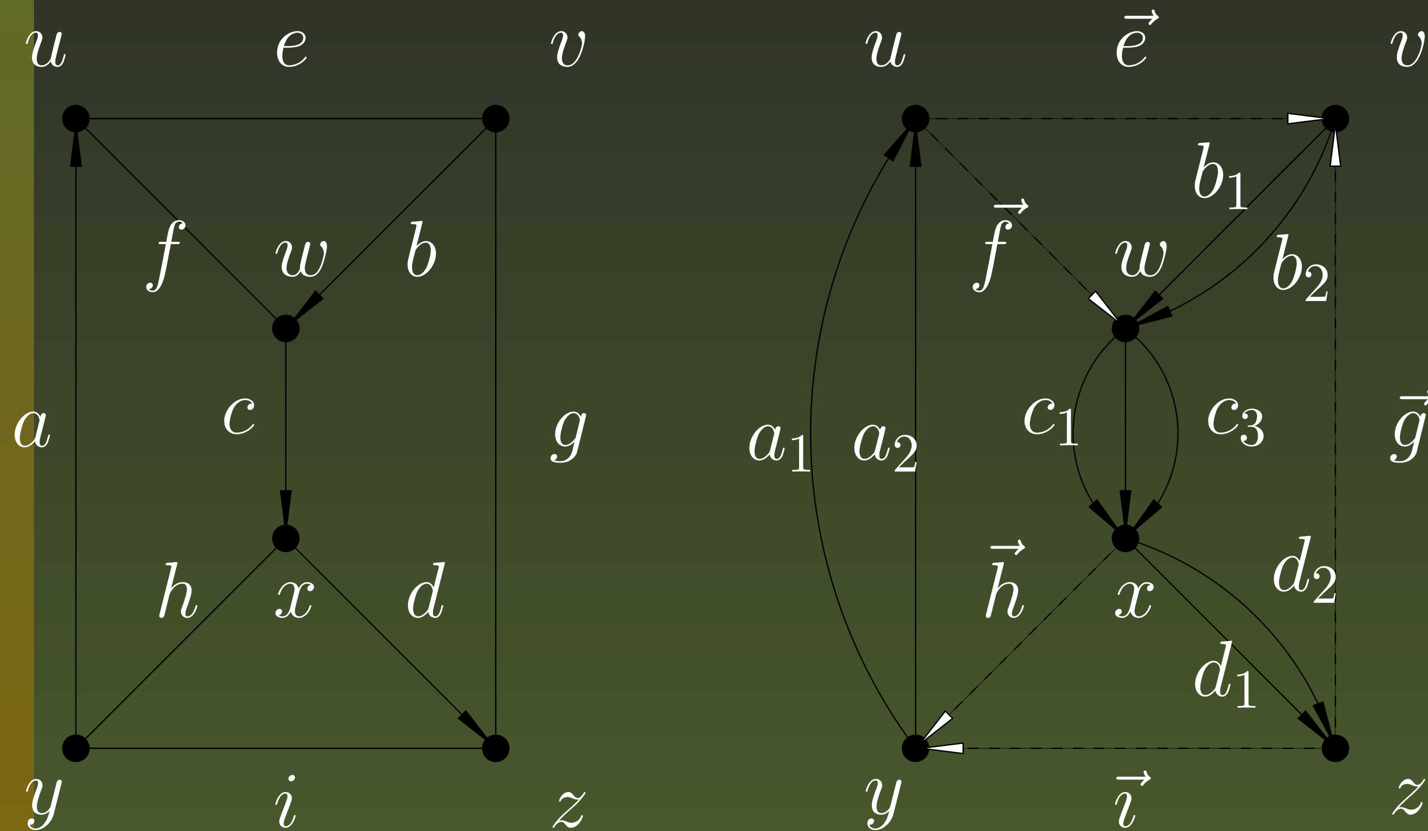

Figure 1: Mixed graph and arcs postman tour

Arcs Postman Problem

- Given a strongly connected mixed graph $M = (V, E, A)$ and a vector $c \in \mathbb{Q}_+^A$, the *cost* of an *arcs postman tour* T of M is the sum of the costs of the arcs traversed by T.
- **Problem:** Obtain the minimum cost MAPT(M, c) of an arcs postman tour of M.

Computational Complexity

- It is NP-complete to decide whether a mixed graph $M = (V, E, A)$ has an arcs postman tour (reduction from not-all-equal satisfiability).
- The arcs postman problem (minimization) can be solved in polynomial time (for example, using linear programming methods) if:
  - M is series-parallel,
  - M is Eulerian, or
  - $D = (V, A)$ is a forest.

Variable Gadget

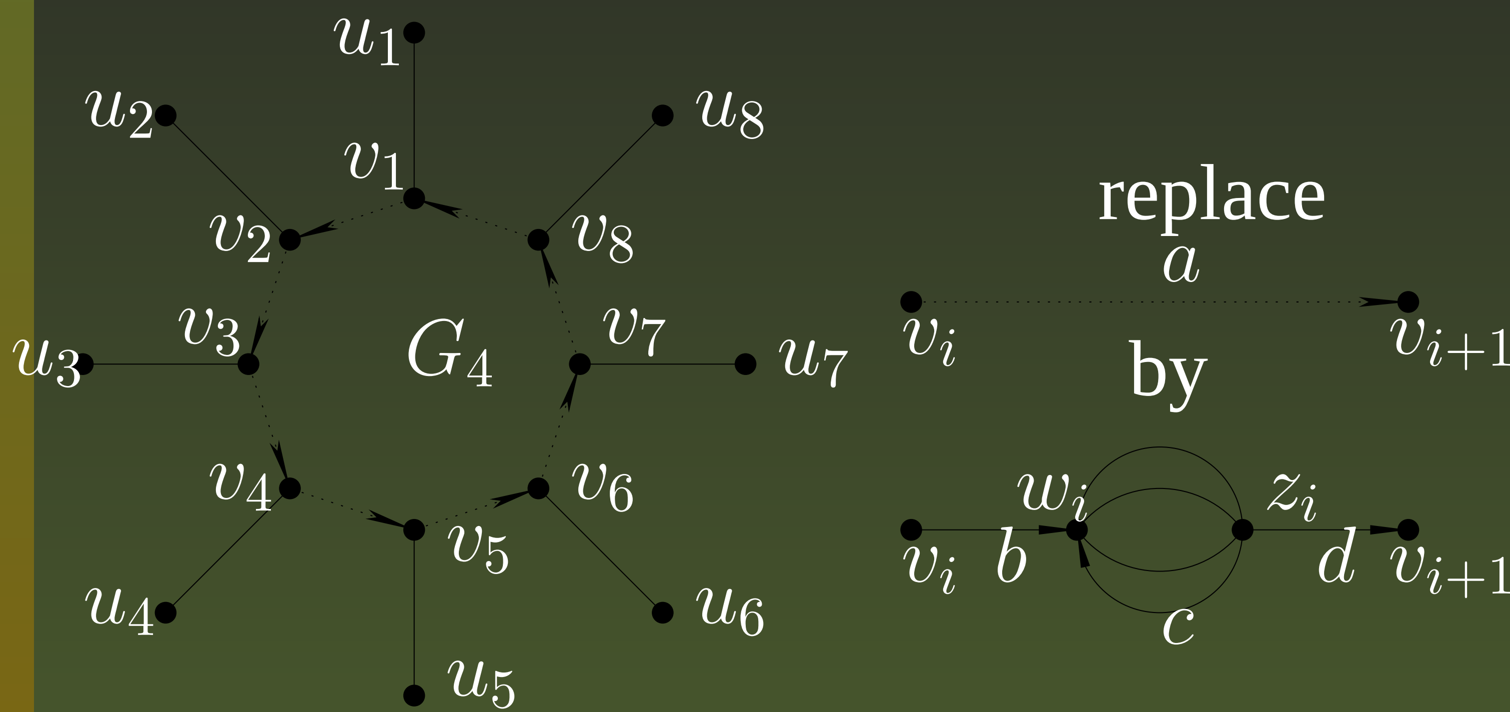

Figure 2: The subgraph G_j for $q_j = 4$.

Clause Gadget

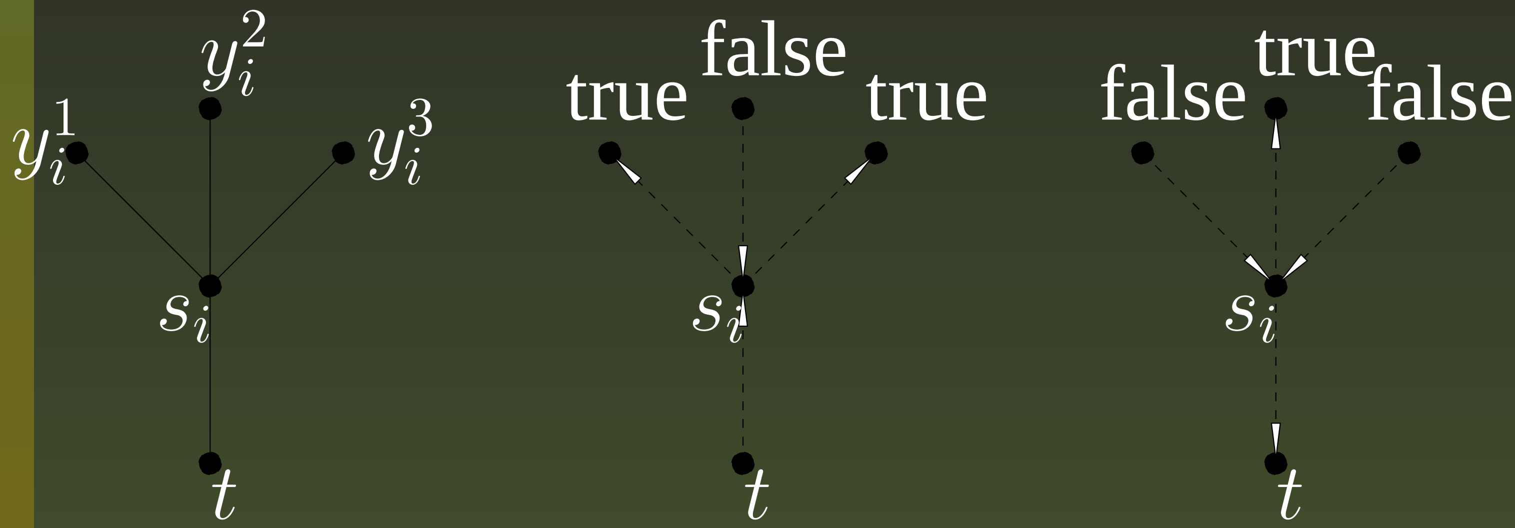

Figure 3: The clause subgraph H_i.

Connecting the Gadgets

all these edges connect to vertex t

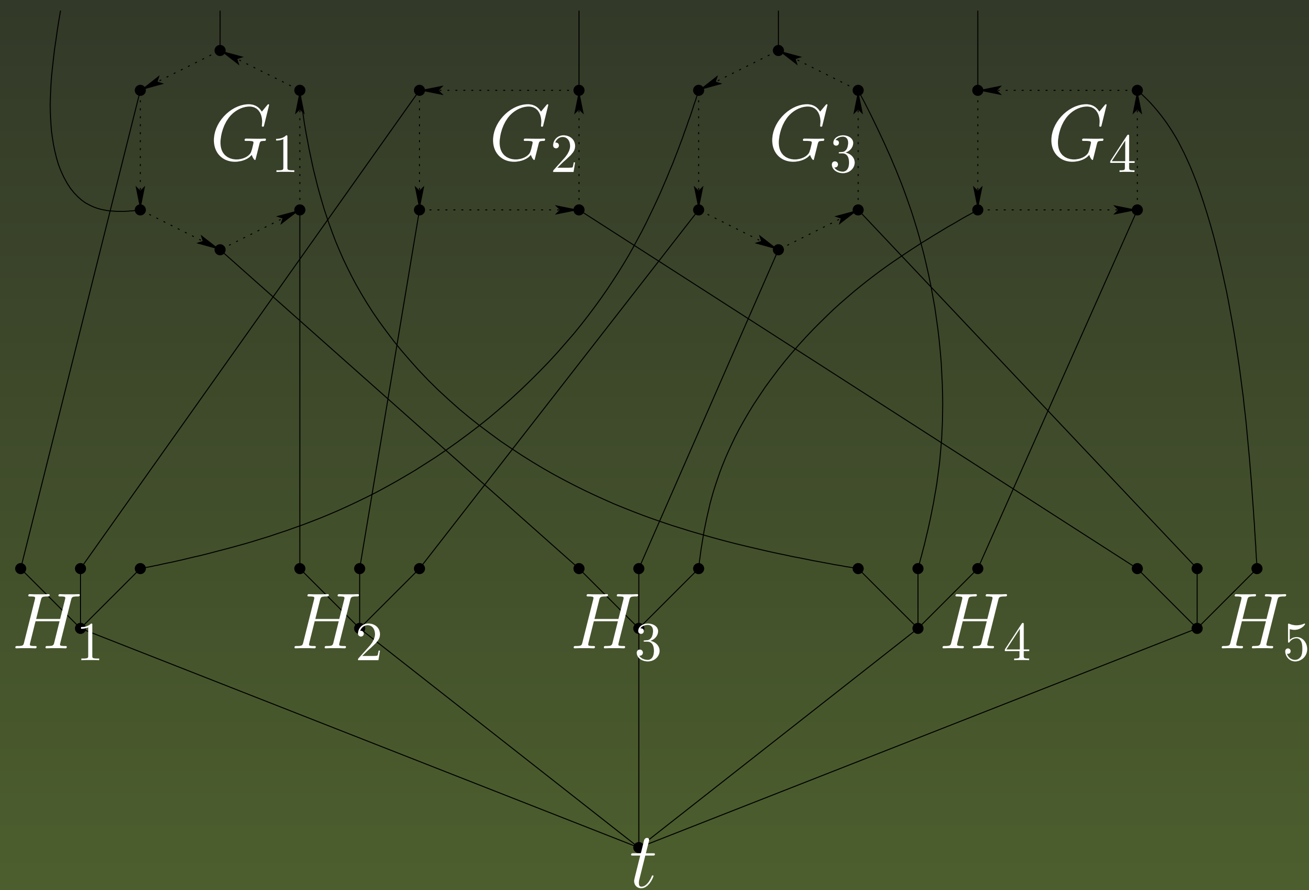

Figure 4: An instance of arcs postman tour.

Necessary Conditions (Veerasamy)

- If $M = (V, E, A)$ has an arcs postman tour then:
  - M is connected,
  - for all *outgoing* $S \subseteq V$ there are more edges *crossing* S than arcs *leaving* S, and
  - for all *undirected* $S \subseteq V$ there is an even number of edges *crossing* S.
- These conditions are independent.
- These conditions can be tested in polynomial time.
- These conditions are not sufficient.

Stronger Necessary Conditions (1)

- For $S \subseteq V$ let $\text{sur}_M(S) = d_E(S) - d_A(S)$.
- Assume M has an arcs postman tour with edges oriented as in I.
- If $S \subseteq V$ is outgoing then

$$d_I(\bar{S}) \geq d_A(S) + d_I(S)$$

and therefore

$$d_E(S) = d_I(S) + d_I(\bar{S}) \geq d_A(S) + 2d_I(S),$$

which implies that

$$d_I(S) \leq \lfloor \tfrac{1}{2}(d_E(S) - d_A(S)) \rfloor = \lfloor \tfrac{1}{2}\text{sur}_M(S) \rfloor.$$

Stronger Necessary Conditions (2)

- Similarly, if $T \subseteq V$ is outgoing then

$$d_I(T) \leq \lfloor \tfrac{1}{2}\text{sur}_M(T) \rfloor.$$

- Adding these two inequalities we get

$$d_I(S) + d_I(T) \leq \lfloor \tfrac{1}{2}\text{sur}_M(S) \rfloor + \lfloor \tfrac{1}{2}\text{sur}_M(T) \rfloor.$$

- Observe that a lower bound for $d_I(S) + d_I(T)$ is the number of edges between $S \setminus T$ and $T \setminus S$. Therefore

$$|E(S \setminus T, T \setminus S)| \leq \lfloor \tfrac{1}{2}\text{sur}_M(S) \rfloor + \lfloor \tfrac{1}{2}\text{sur}_M(T) \rfloor.$$

Stronger Necessary Conditions (3)

- Our condition is stronger and generalizes the *outgoing* and *undirected* conditions of Veerasamy.
- Our condition can also be tested in polynomial time (using *minimum-odd-cut* and *flow* subroutines).
- Our condition is not sufficient:

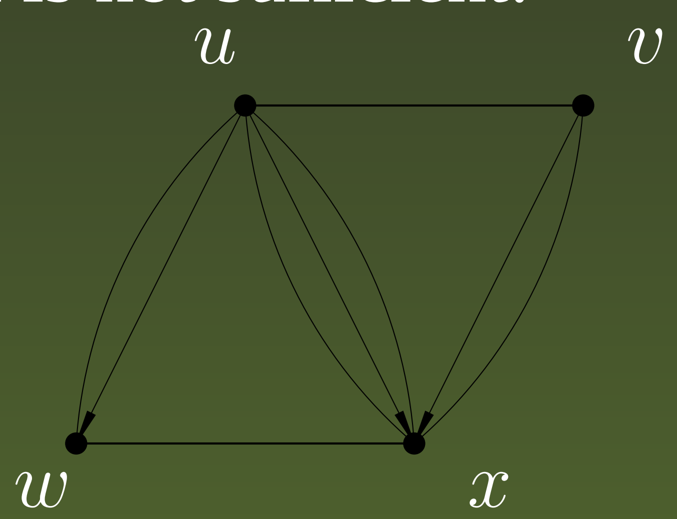

Figure 5: A graph with no arcs postman tour.

Conclusions and Further Work

- Strenghten the complexity results for the arcs postman problem (in both minimization and feasibility versions).
- Find classes of mixed graphs for which there are approximation algorithms for the arcs postman problem.
- Find (faster) algorithms to test our necessary condition(s).
- Find classes of mixed graphs for which our necessary condition is also sufficient.