The Windy Postman Problem on Series-Parallel Graphs

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The Windy Postman Problem

- Let $G = (V, E)$ be a connected, undirected graph, let $\vec{G} = (V, \vec{E})$ be its associated directed graph, and let $c \in \mathbb{Z}_+^{\vec{E}}$.
- The *windy postman problem* is to find the minimum cost of a tour of G traversing all its edges, where the cost of traversal of an edge depends on the direction.
- Theorem (Guan, 1984): This problem is NP-hard, even if restricted to planar G.

Integer Programming Formulation

- Let $P(G)$ be the convex hull of the feasible solutions of the integer program:

$$\begin{array}{rcl} \text{WPP}(G,c) & = & \min c^\top x \\ \text{subject to} & & \\ x(\vec{\delta}_G(\bar{v})) - x(\vec{\delta}_G(v)) & = & 0 \text{ for all } v \in V \\ x_{e^+} + x_{e^-} & \geq & 1 \text{ for all } e \in E \\ x_{e^+}, x_{e^-} & \geq & 0 \text{ for all } e \in E \\ x_{e^+}, x_{e^-} & & \text{integral for all } e \in E. \end{array}$$

Linear Programming Relaxation

- Let $Q(G)$ be the region defined by:

$$\begin{aligned} x(\vec{\delta}_G(\bar{v})) - x(\vec{\delta}_G(v)) &= 0 \text{ for all } v \in V \\ x_{e^+} + x_{e^-} &\geq 1 \text{ for all } e \in E \\ x_{e^+}, x_{e^-} &\geq 0 \text{ for all } e \in E. \end{aligned}$$

- Theorem (Win, 1987): Let G be connected. $P(G) = Q(G)$ if and only if G is *Eulerian*.

Odd-Cut Constraints

- Let $S \subseteq V$ be such that $|\delta_G(S)|$ is odd. Then at least one element of $\delta_G(S)$ must be used more than once.
- Therefore the following is valid for $P(G)$:

$$x(\vec{\delta}_G(S)) + x(\vec{\delta}_G(\bar{S})) \geq |\delta_G(S)| + 1.$$

- Odd-cut constraints were introduced by Edmonds and Johnson (1973).

Windy Postman Perfect Graphs

- Let $O(G)$ be the subset of $Q(G)$ that satisfies all the odd-cut constraints.
- We say that G is *windy postman perfect* if the polyhedron $O(G)$ is integral. Equivalently, G is windy postman perfect if $O(G) = P(G)$.
- Theorem (Grötschel and Win, 1992): There exists a polynomial-time algorithm to solve the windy postman problem for the class of windy postman perfect graphs.

More on WPP Graphs

- Trees and Eulerian graphs are windy postman perfect (Win, 1987).
- K_4 is not windy postman perfect, $K_{3,3}$ is.
- Windy postman perfection is not closed under edge deletion.
- But it is closed under edge subdivision, edge contraction, vertex identification, and deletion of *two* parallel edges.
- Win conjectured that series-parallel graphs are windy postman perfect.

Windy Postman Ideal Graphs

- Let $l \in \mathbb{Z}_+^E$, and let $b \in \mathbb{Z}^V$ with $b(V) = 0$. We say that $S \subseteq V$ is *odd* if $b(S) + l(\delta_G(S))$ is odd. Let $O(G, l, b)$ be given by

$$
\begin{aligned}
x(\vec{\delta}_G(\bar{v})) - x(\vec{\delta}_G(v)) &= b_v \text{ for all } v \in V \\
x_{e^+} + x_{e^-} &\geq l_e \text{ for all } e \in E \\
x(\vec{\delta}_G(S)) + x(\vec{\delta}_G(\bar{S})) &\geq l(\delta_G(S)) + 1 \text{ for odd } S \\
x_{e^+}, x_{e^-} &\geq 0 \text{ for all } e \in E.
\end{aligned}
$$

- We say that G is *windy postman ideal* if $O(G, l, b)$ is integral for all choices of l and b.

Characterization of WPI Graphs

- Theorem: Windy postman ideality is closed under taking graph minors.
- Theorem: The following are equivalent:
  1. G is series-parallel.
  2. G is windy postman ideal.
  3. $O(G, l, 0)$ is integral for all $l \in \{0, 1\}^E$.
- Corollary: Series-parallel graphs are windy postman perfect.

Conclusions and Further Work

- We have extended the class of known windy postman perfect graphs.
- We are searching for a *combinatorial* algorithm for the windy postman problem on windy postman perfect graphs.
- We are studying windy postman perfection of *grafts*. We have shown that this property is closed under taking graft minors and we have found two excluded minors.