Feasibility of the Mixed Postman Problem with Restrictions on the Edges

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The Mixed Postman Problem

- Let $M = (V, E, A)$ be a strongly connected mixed graph and let $c \in \mathbb{Z}_+^{E \cup A}$.
- The *mixed postman problem* is to find the minimum cost of a tour of M traversing all of its edges and arcs.
- This problem is NP-hard, even if restricted to planar inputs.
- Feasibility can be decided in polynomial time.

Restrictions on the Edges

We are interested on tours that use edges once.

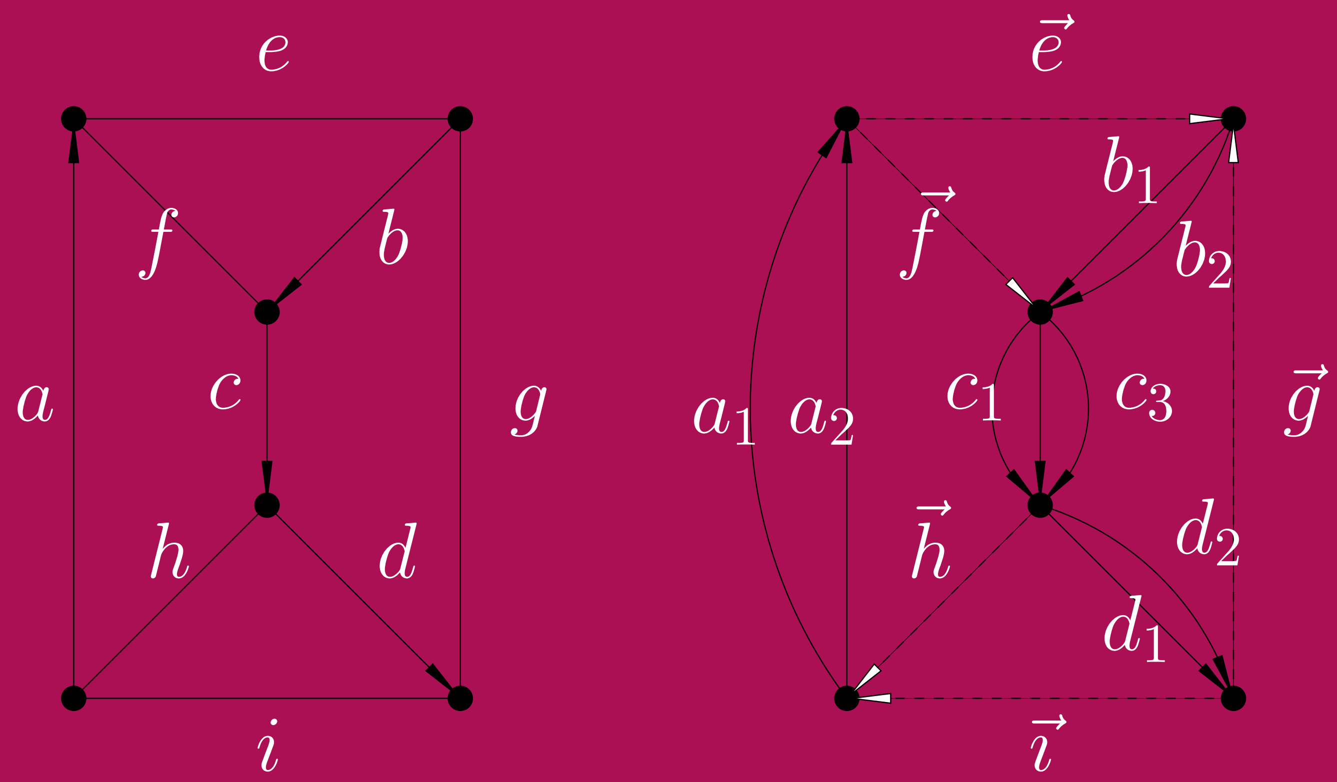


Figure 1: An arcs postman tour.

Complexity of Feasibility

- Feasibility is NP-complete.
- It remains NP-complete for planar inputs.
- Feasibility can be decided in poly-time if
  - M has total degree even at every vertex,
  - M is series-parallel, or
  - A is a forest.
- Optimization in poly-time too.
- We are interested in necessary conditions (which can be verified in poly-time).

Even Set Condition

- For $S \subseteq V$, let $\delta_M(S)$ be the set of edges and arcs between S and $\bar{S} = V \setminus S$.
- We say that S is *undirected* if $\delta_M(S) \subseteq E$.
- If M has an arcs postman tour and S is undirected then $|\delta_M(S)|$ must be even.
- This condition (due to Veerasamy) can be verified in polynomial time.

Outgoing Set Condition

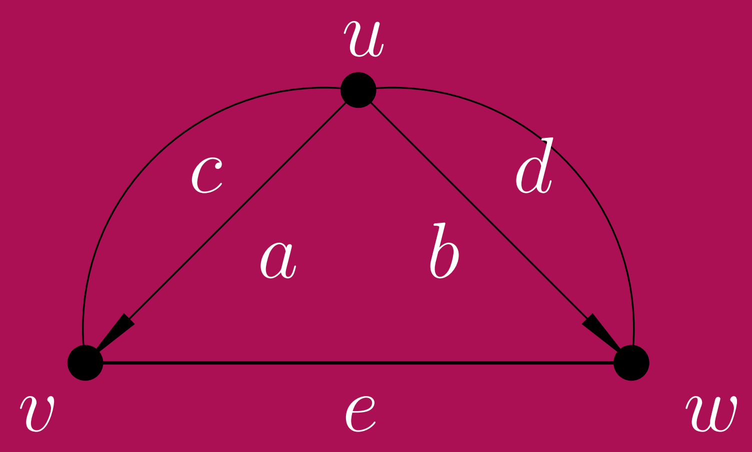
- We say that S is *outgoing* if all arcs in $\delta_M(S)$ go from S to $\bar{S}$.
- If M has an arcs postman tour and S is outgoing then

$$|\delta_M(S) \cap E| \geq |\delta_M(S) \cap A|.$$

- This condition (also due to Veerasamy) can be verified in polynomial time.

Independency and Sufficiency

- K_2 satisfies the *outgoing set* condition but *not* the even set condition.
- $\vec{K}_2$ satisfies the *even set* condition but *not* the outgoing set condition.
- These conditions are not sufficient:



Two Outgoing Sets Condition

- For outgoing S we define the *surplus* of S as

$$\text{sur}_M(S) = |\delta_M(S) \cap E| - |\delta_M(S) \cap A|.$$

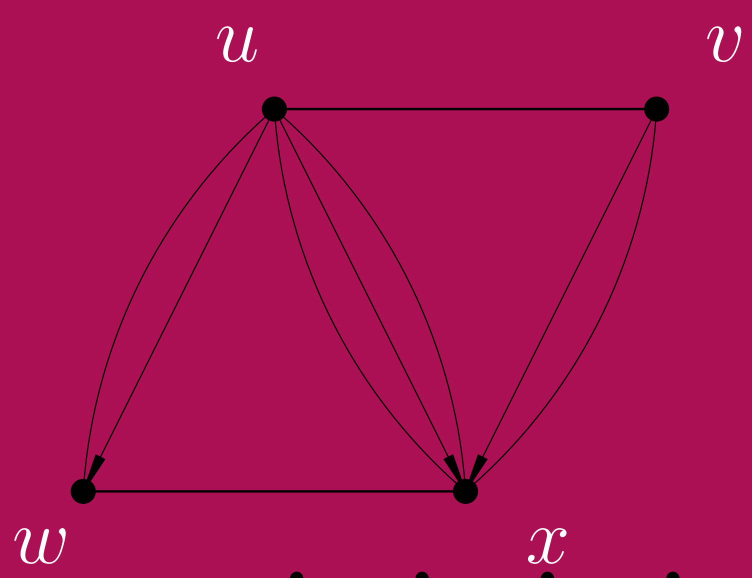
- **Theorem:** Let M have an arcs postman tour and let S, T be outgoing. Then

$$|E(S \setminus T, T \setminus S)| \leq \lfloor \frac{1}{2} \text{sur}_M(S) \rfloor + \lfloor \frac{1}{2} \text{sur}_M(T) \rfloor.$$

- This condition can be verified in polynomial time (nicer algorithm for planar M).

Strength and Sufficiency

- Our condition implies and is stronger than
  - the even set condition (take $T = \bar{S}$) and
  - the outgoing set condition (take $T = S$).
- Our condition is still not sufficient:



Many Outgoing Sets

- Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be k outgoing subsets:

$$|E_k^{\mathcal{S}}| \leq \sum_{i=1}^{k} \lfloor \frac{1}{2} \text{sur}_M(S_i) \rfloor$$

where $E_k^{\mathcal{S}}$ is a *certain* subset of edges.

- **Conjecture:** There exists $f : \mathbb{N} \rightarrow \mathbb{N}$ for which: A connected mixed graph M has an arcs postman tour iff it satisfies the k-outgoing-sets condition for all $1 \leq k \leq f(|V(M)|)$.

f is Linear or Worse

The mixed graph M_n satisfies the k-outgoing-sets condition for each $1 \leq k \leq 2n-1$, but does not satisfy the $2n$-outgoing-sets condition.

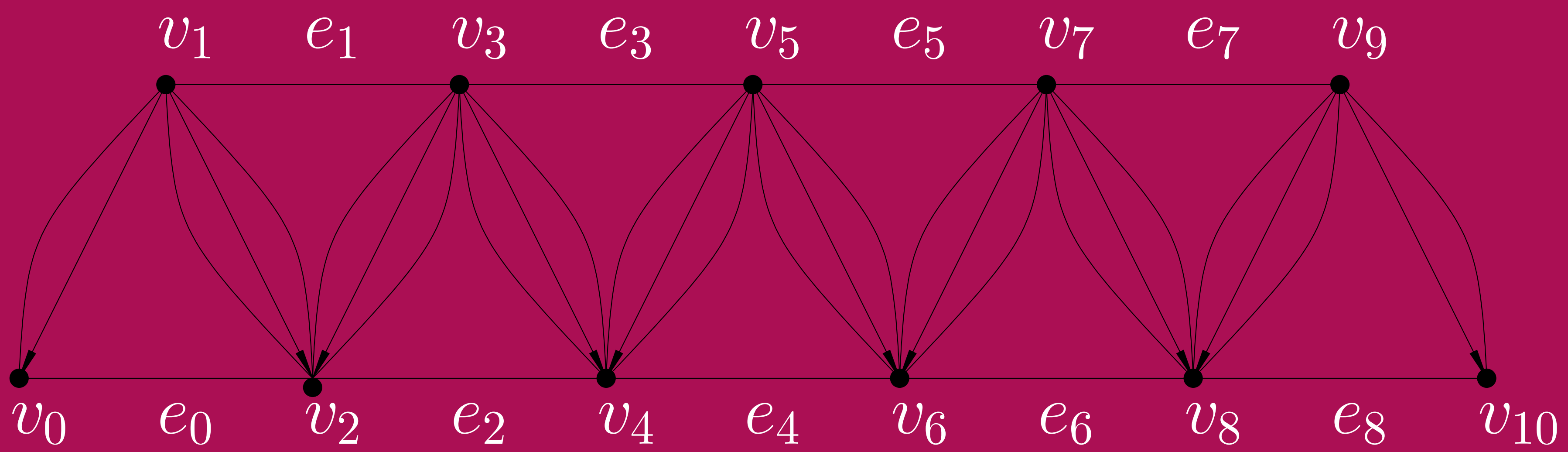


Figure 2: The mixed graph M_5.

Characterization

The following statements are equivalent:

- M has an arcs postman tour for every orientation of its edges.
- The only outgoing sets of M are $\emptyset$ and V.
- $D = (V, A)$ is strongly connected.
- For all $k \in \mathbb{N}$, and for all families $\mathcal{S}$ of k outgoing sets, M satisfies:

$$\sum_{e \in E} m^{+}_{\mathcal{S}}(e) \leq \sum_{i=1}^{k} \lfloor \frac{1}{2} \mathrm{sur}_M(S_i) \rfloor .$$

Conclusions

- New necessary conditions for feasibility.
- Some of them can be verified in poly-time.
- Can we verify some others too?
- Are our conditions sufficient?
- We are searching for other classes of mixed graphs for which we can decide feasibility or optimize in polynomial time.