Solving the Square Path Problem up to 20×20

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Contents

- The Square Path Problem
- Theoretical Results
- Computational Results
- Conclusions and Further Work

The Square Path Problem

- Consider an $m \times n$ rectangular lattice with points colored either black or white.
- A *square path* is a closed path in the shape of a square with edges parallel to those of the lattice.
- The *square path problem* is to find the minimum number $f(m, n)$ of black points needed in the rectangular lattice so that every square path has at least one black point on it.
- This problem was posed by Morris (1998) for square lattices (that is, for $m = n$.)

Examples

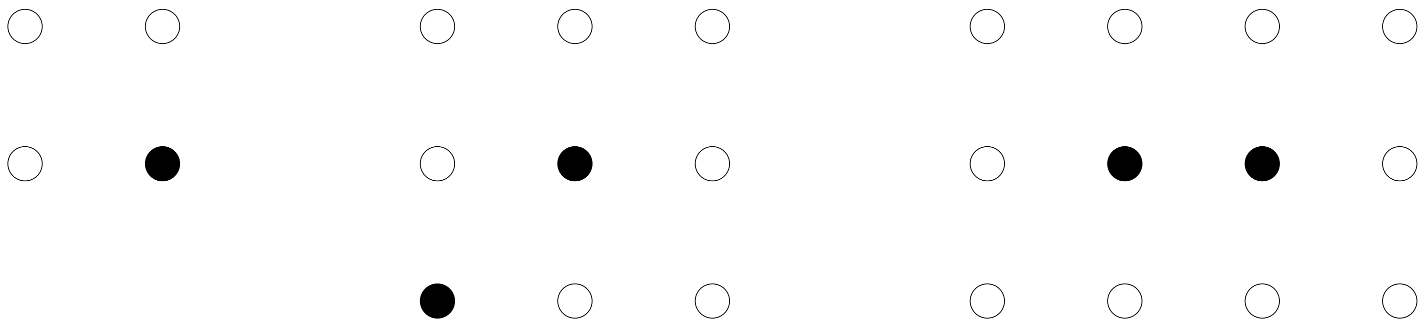

Figure 1: $f(2,2) = 1$, $f(3,3) = 2$, $f(3,4) = 2$

Partition Lemma

(Williams 2000) Assume that we have partitioned an $m \times n$ rectangular lattice into k lattices of sizes $m_1 \times n_1, \ldots, m_k \times n_k$. Then

$$f(m,n) \geq \sum_{i=1}^{k} f(m_i, n_i).$$

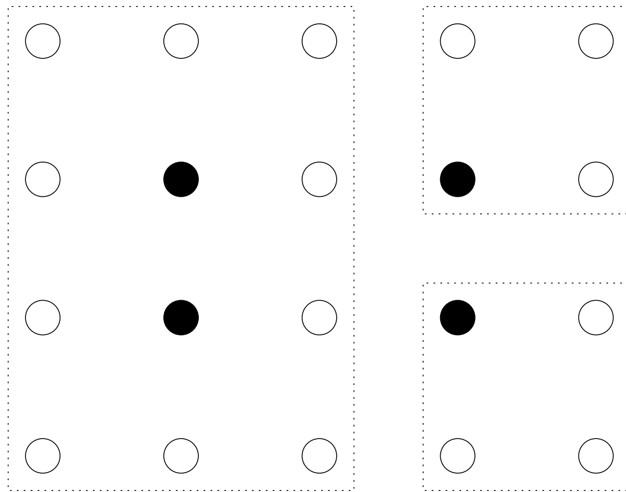

Figure 2: $4 \geq f(4,5) \geq f(4,3) + f(2,2) + f(2,2)$

What was known?

- Williams (2000) computed $f(m,n)$ for all $m,n \leq 6$.
- Using CPLEX, Williams (2000) computed upper bounds for $f(n,n)$ for all $n \leq 14$.
- Dean and Zuniga (2004) computed the exact value of $f(m,n)$ for all $m \leq 4$ and all $n \geq 1$.

Computational results

- Williams (2000) computed $f(n,n)$ for all $n \leq 14$ in about 14 hours. The computation of $f(15,15)$ aborted after another 48 hours.
- We developed a backtracking algorithm (with some improvements based on dynamic programming) that computed $f(m,n)$ for all $m,n \leq 20$ in 62 hours.
- In particular, our algorithm computed $f(m,n)$ for all $m,n \leq 15$ in 43 seconds!

The main idea (I)

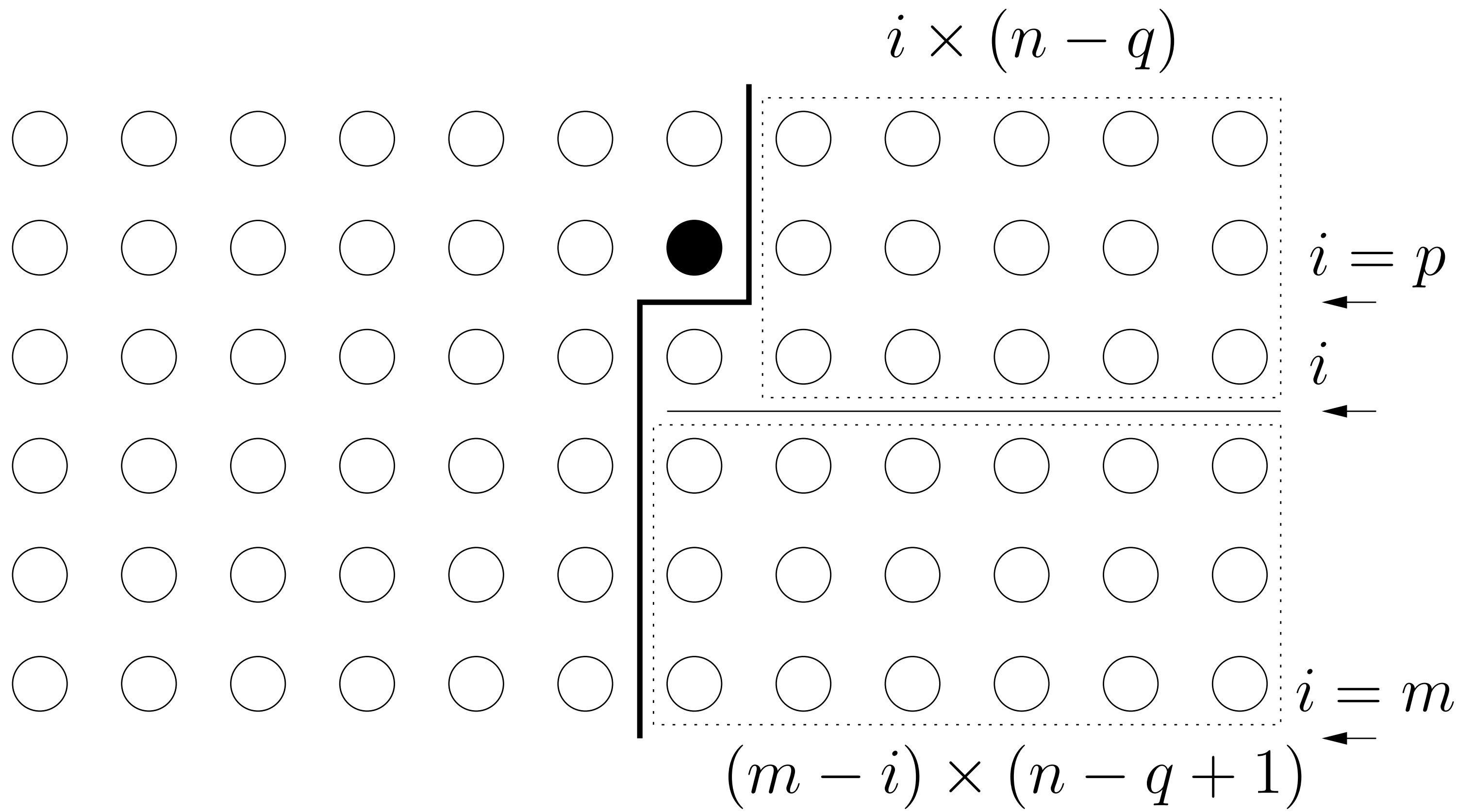

Figure 3: Horizontal partition.

The main idea (II)

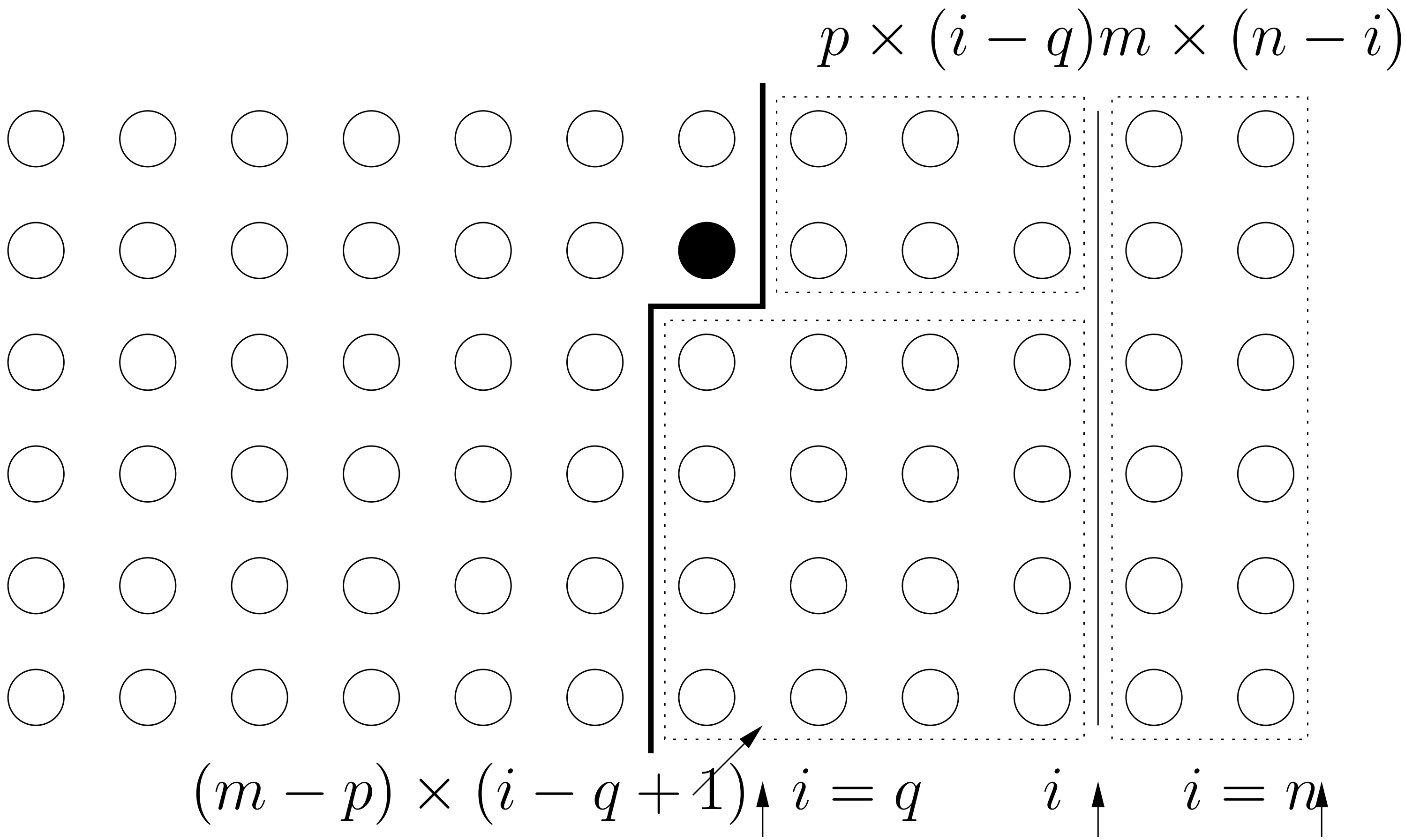

Figure 4: Vertical partition.

Theoretical results

- We proved that $f(5, n) = \lfloor (4n-2)/3 \rfloor$ for $n \geq 1$.
- We also proved that $f(6, n) = \lfloor (3n)/2 \rfloor$ for $n \geq 2$.
- Our proofs are by induction and they use a structural characterization of optimal solutions.

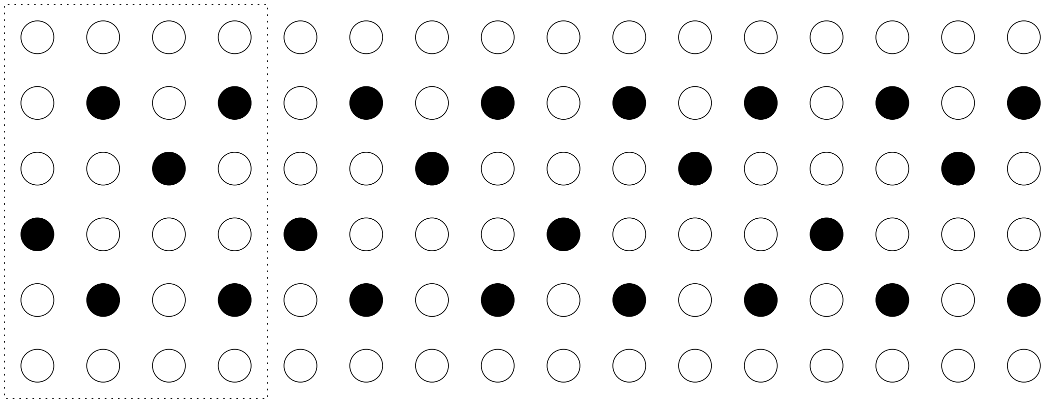

Figure 5: $6 \times n$ rectangular grid.

Conclusions and Further Work

- We have conjectures on the exact values of $f(7, n)$, $f(8, n)$, $f(9, n)$, and $f(10, n)$ for all $n \geq 1$.
- We are interested on the *rectangle path problem* and the *cycle path problem*.
- We are also interested on the related problems on cylindrical, toroidal, projective, Moebius, and Klein bottle grids.