elements:[15]

$$\hat{A} \xrightarrow[|a_1\rangle\text{-}|a_2\rangle \text{ basis}]{} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \tag{2.147}$$

All of the results (2.132) through (2.147) can be extended in a straightforward fashion to larger dimensional bases, as introduced in Section 1.6. For example, the identity operator is given by $\sum_n |a_n\rangle\langle a_n|$ in the more general case.

PROBLEMS

2.1. Show that

$$\lim_{N\to\infty}\left(1 + \frac{x}{N}\right)^N = e^x$$

by comparing the Taylor series expansions for the two functions.

2.2. Use Dirac notation (the properties of kets, bras, and inner products) directly without explicitly using matrix representations to establish that the projection operator $\hat{P}_+$ is Hermitian. Use the fact that $\hat{P}_+^2 = \hat{P}_+$ to establish that the eigenvalues of the projection operator are 1 and 0.

2.3. Determine the matrix representation of the rotation operator $\hat{R}(\phi\mathbf{k})$ using the states $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$ as a basis. Using your matrix representation, verify that the rotation operator is unitary, that is, that it satisfies $\hat{R}^\dagger(\phi\mathbf{k})\hat{R}(\phi\mathbf{k}) = 1$.

2.4. Determine the column vectors representing the states $|+\mathbf{x}\rangle$ and $|-\mathbf{x}\rangle$ using the states $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ as a basis.

2.5. What is the matrix representation of $\hat{J}_z$ using the states $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ as a basis? Use this representation to evaluate the expectation value of S_z for a collection of particles each in the state $|-\mathbf{y}\rangle$.

2.6. Evaluate $\hat{R}(\theta\mathbf{j})|+\mathbf{z}\rangle$, where $\hat{R}(\theta\mathbf{j}) = e^{-i\hat{J}_y\theta/\hbar}$ is the operator that rotates kets counterclockwise by angle θ about the y axis. Show that $\hat{R}(\frac{\pi}{2}\mathbf{j})|+\mathbf{z}\rangle = |+\mathbf{x}\rangle$. *Suggestion:* Express the ket $|+\mathbf{z}\rangle$ as a superposition of the kets $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ and take advantage of the fact that $\hat{J}_y|\pm\mathbf{y}\rangle = (\pm\hbar/2)|\pm\mathbf{y}\rangle$; then switch back to the $|+\mathbf{z}\rangle$-$|-\mathbf{z}\rangle$ basis.

2.7. Work out the matrix representations of the projection operators $\hat{P}_+ = |+\mathbf{z}\rangle\langle+\mathbf{z}|$ and $\hat{P}_- = |-\mathbf{z}\rangle\langle-\mathbf{z}|$ using the states $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ of a spin-$\frac{1}{2}$ particle as a basis. Check that the results (2.51) and (2.52) are satisfied using these matrix representations.

2.8. A photon polarization state for a photon propagating in the z direction is given by

$$|\psi\rangle = \sqrt{\frac{2}{3}}|x\rangle + \frac{i}{\sqrt{3}}|y\rangle$$

[15] In general, there are an infinite number of sets of basis states that may be used to form representations in matrix mechanics. For example, in addition to the states $|\pm\mathbf{z}\rangle$, the states $|\pm\mathbf{x}\rangle$ can be used as a basis to represent states and operators for spin-$\frac{1}{2}$ particles. However, since $|\pm\mathbf{x}\rangle$ are not eigenstates of $\hat{J}_z$, the matrix representation of this operator using these states as a basis is not diagonal, as (2.96) shows.

and

$$\hat{P}_+|-\mathbf{z}\rangle = |+\mathbf{z}\rangle\langle+\mathbf{z}|-\mathbf{z}\rangle = 0 \tag{2.49b}$$

Thus $|+\mathbf{z}\rangle$ is an eigenstate of the projection operator $\hat{P}_+$ with eigenvalue 1, and $|-\mathbf{z}\rangle$ is an eigenstate of the projection operator $\hat{P}_+$ with eigenvalue 0. We can obtain a physical realization of the projection operator $\hat{P}_+$ from the modified SG device by blocking the path that would be taken by a particle in the state $|-\mathbf{z}\rangle$, that is, by blocking the lower path, as shown in Fig. 2.4*b*. Each particle in the state $|+\mathbf{z}\rangle$ entering the device exits the device. We can then say we have obtained the eigenvalue 1. Since none of the particles in the state $|-\mathbf{z}\rangle$ that enters the device also exits the device, we can say we have obtained the eigenvalue 0 in this case.

Similarly, we can create a physical realization of the projection operator $\hat{P}_-$ by blocking the upper path in the modified SG device, as shown in Fig. 2.4*c*. Then each particle in the state $|-\mathbf{z}\rangle$ that enters the device also exits the device:

$$\hat{P}_-|-\mathbf{z}\rangle = |-\mathbf{z}\rangle\langle-\mathbf{z}|-\mathbf{z}\rangle = |-\mathbf{z}\rangle \tag{2.50a}$$

while none of the particles in the state $|+\mathbf{z}\rangle$ exits the device:

$$\hat{P}_-|+\mathbf{z}\rangle = |-\mathbf{z}\rangle\langle-\mathbf{z}|+\mathbf{z}\rangle = 0 \tag{2.50b}$$

Hence the eigenvalues of $\hat{P}_-$ are 1 and 0 for the states $|-\mathbf{z}\rangle$ and $|+\mathbf{z}\rangle$, respectively.

Notice that each of the particles that has traversed one of the projection devices is certain to pass through a subsequent projection device of the same type:

$$\begin{aligned}\hat{P}_+^2 &= (|+\mathbf{z}\rangle\langle+\mathbf{z}|)(|+\mathbf{z}\rangle\langle+\mathbf{z}|)\\ &= |+\mathbf{z}\rangle\langle+\mathbf{z}|+\mathbf{z}\rangle\langle+\mathbf{z}| = |+\mathbf{z}\rangle\langle+\mathbf{z}| = \hat{P}_+ \qquad (2.51a)\\ \hat{P}_-^2 &= (|-\mathbf{z}\rangle\langle-\mathbf{z}|)(|-\mathbf{z}\rangle\langle-\mathbf{z}|)\\ &= |-\mathbf{z}\rangle\langle-\mathbf{z}|-\mathbf{z}\rangle\langle-\mathbf{z}| = |-\mathbf{z}\rangle\langle-\mathbf{z}| = \hat{P}_- \qquad (2.51b)\end{aligned}$$

while a particle that passes a first projection device will surely fail to pass a subsequent projection device of the opposite type:

$$\begin{aligned}\hat{P}_+\hat{P}_- &= (|+\mathbf{z}\rangle\langle+\mathbf{z}|)(|-\mathbf{z}\rangle\langle-\mathbf{z}|)\\ &= |+\mathbf{z}\rangle\langle+\mathbf{z}|-\mathbf{z}\rangle\langle-\mathbf{z}| = 0 \qquad (2.52a)\\ \hat{P}_-\hat{P}_+ &= (|-\mathbf{z}\rangle\langle-\mathbf{z}|)(|+\mathbf{z}\rangle\langle+\mathbf{z}|)\\ &= |-\mathbf{z}\rangle\langle-\mathbf{z}|+\mathbf{z}\rangle\langle+\mathbf{z}| = 0 \qquad (2.52b)\end{aligned}$$

These results are illustrated in Fig. 2.5.

Our discussion of the identity operator and the projection operators has arbitrarily been phrased in terms of the S_z basis. We could as easily have expressed the same state $|\psi\rangle$ in terms of the S_x basis as $|\psi\rangle = |+\mathbf{x}\rangle\langle+\mathbf{x}|\psi\rangle + |-\mathbf{x}\rangle\langle-\mathbf{x}|\psi\rangle$. Thus we can also express the identity operator as

$$|+\mathbf{x}\rangle\langle+\mathbf{x}| + |-\mathbf{x}\rangle\langle-\mathbf{x}| = 1 \tag{2.53}$$