elements:[15]

$$\hat{A} \xrightarrow[|a_1\rangle\text{-}|a_2\rangle \text{ basis}]{} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \tag{2.147}$$

All of the results (2.132) through (2.147) can be extended in a straightforward fashion to larger dimensional bases, as introduced in Section 1.6. For example, the identity operator is given by $\sum_n |a_n\rangle\langle a_n|$ in the more general case.

PROBLEMS

2.1. Show that

$$\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = e^x$$

by comparing the Taylor series expansions for the two functions.

2.2. Use Dirac notation (the properties of kets, bras, and inner products) directly without explicitly using matrix representations to establish that the projection operator $\hat{P}_+$ is Hermitian. Use the fact that $\hat{P}_+^2 = \hat{P}_+$ to establish that the eigenvalues of the projection operator are 1 and 0.

2.3. Determine the matrix representation of the rotation operator $\hat{R}(\phi\mathbf{k})$ using the states $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$ as a basis. Using your matrix representation, verify that the rotation operator is unitary, that is, that it satisfies $\hat{R}^\dagger(\phi\mathbf{k})\hat{R}(\phi\mathbf{k}) = 1$.

2.4. Determine the column vectors representing the states $|+\mathbf{x}\rangle$ and $|-\mathbf{x}\rangle$ using the states $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ as a basis.

2.5. What is the matrix representation of $\hat{J}_z$ using the states $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ as a basis? Use this representation to evaluate the expectation value of S_z for a collection of particles each in the state $|-\mathbf{y}\rangle$.

2.6. Evaluate $\hat{R}(\theta\mathbf{j})|+\mathbf{z}\rangle$, where $\hat{R}(\theta\mathbf{j}) = e^{-i\hat{J}_y\theta/\hbar}$ is the operator that rotates kets counterclockwise by angle θ about the y axis. Show that $\hat{R}(\frac{\pi}{2}\mathbf{j})|+\mathbf{z}\rangle = |+\mathbf{x}\rangle$. *Suggestion:* Express the ket $|+\mathbf{z}\rangle$ as a superposition of the kets $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ and take advantage of the fact that $\hat{J}_y|\pm\mathbf{y}\rangle = (\pm\hbar/2)|\pm\mathbf{y}\rangle$; then switch back to the $|+\mathbf{z}\rangle$-$|-\mathbf{z}\rangle$ basis.

2.7. Work out the matrix representations of the projection operators $\hat{P}_+ = |+\mathbf{z}\rangle\langle+\mathbf{z}|$ and $\hat{P}_- = |-\mathbf{z}\rangle\langle-\mathbf{z}|$ using the states $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ of a spin-$\frac{1}{2}$ particle as a basis. Check that the results (2.51) and (2.52) are satisfied using these matrix representations.

2.8. A photon polarization state for a photon propagating in the z direction is given by

$$|\psi\rangle = \sqrt{\frac{2}{3}}|x\rangle + \frac{i}{\sqrt{3}}|y\rangle$$

[15] In general, there are an infinite number of sets of basis states that may be used to form representations in matrix mechanics. For example, in addition to the states $|\pm\mathbf{z}\rangle$, the states $|\pm\mathbf{x}\rangle$ can be used as a basis to represent states and operators for spin-$\frac{1}{2}$ particles. However, since $|\pm\mathbf{x}\rangle$ are not eigenstates of $\hat{J}_z$, the matrix representation of this operator using these states as a basis is not diagonal, as (2.96) shows.

(a) What is the probability that a photon in this state will pass through an ideal polarizer with its transmission axis oriented in the y direction?

(b) What is the probability that a photon in this state will pass through an ideal polarizer with its transmission axis y' making an angle ϕ with the y axis?

(c) A beam carrying N photons per second, each in the state $|\psi\rangle$, is totally absorbed by a black disk with its normal to the surface in the z direction. How large is the torque exerted on the disk? In which direction does the disk rotate? *Reminder:* The photon states $|R\rangle$ and $|L\rangle$ each carry a unit $\hbar$ of angular momentum parallel and antiparallel, respectively, to the direction of propagation of the photons.

(d) How would the result for each of these questions differ if the polarization state were

$$|\psi'\rangle = \sqrt{\frac{2}{3}}|x\rangle + \frac{1}{\sqrt{3}}|y\rangle$$

that is, the "i" in the state $|\psi\rangle$ is absent?

2.9. A system of N ideal linear polarizers is arranged in sequence, as shown in Fig. 2.13. The transmission axis of the first polarizer makes an angle of ϕ/N with the y axis. The transmission axis of every other polarizer makes an angle of ϕ/N with respect to the axis of the preceding one. Thus, the transmission axis of the final polarizer makes an angle ϕ with the y axis. A beam of y-polarized photons is incident on the first polarizer.

(a) What is the probability that an incident photon is transmitted by the array?

(b) Evaluate the probability of transmission in the limit of large N.

(c) Consider the special case with the angle $\phi = 90°$. Explain why your result is not in conflict with the fact that $\langle x|y\rangle = 0$.[16]

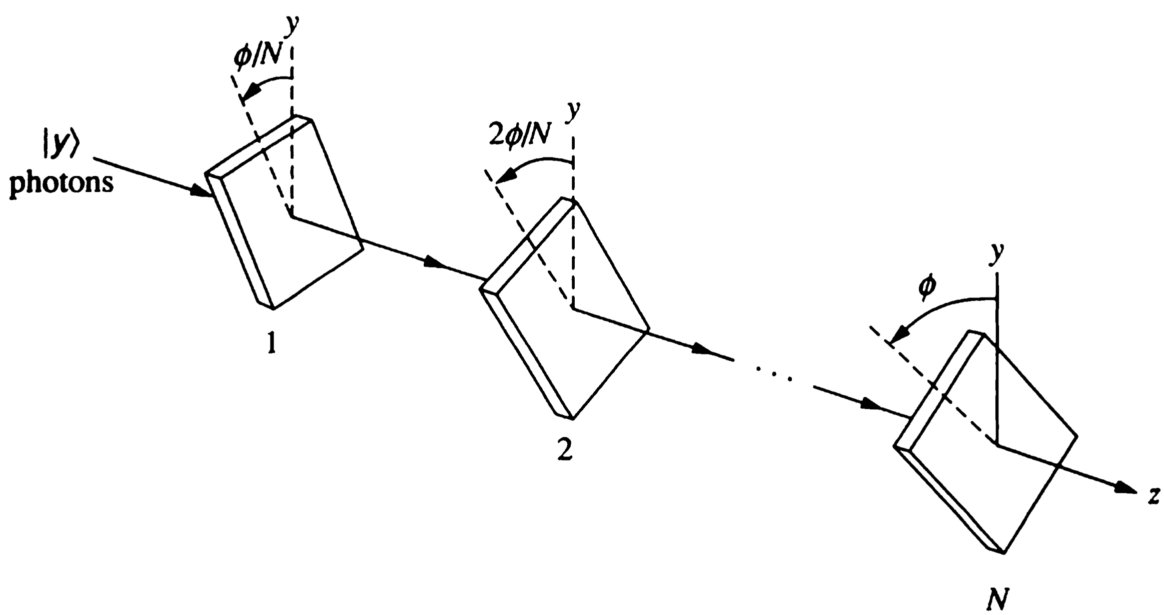

FIGURE 2.13

[16] A nice discussion of the quantum state using photon polarization states as a basis is given by A. P. French and E. F. Taylor, *An Introduction to Quantum Physics*, Norton, New York, 1978, Chapters 6 and 7. Problem 2.9 is adapted from this source.

If the z component of the angular momentum has a definite nonzero value, making the right-hand side of (3.130) nonzero, then we cannot specify either the x or y component of the angular momentum with certainty, because this would require the left-hand side of (3.130) to vanish, in contradiction to the inequality. This uncertainty relation is, of course, built into our results (3.123) and (3.124), which, like (3.130), follow directly from the commutation relations (3.116). Nonetheless, uncertainty relations such as (3.130) bring to the fore the sharp differences between the quantum and the classical worlds. In Chapter 6 we will see how (3.128) and (3.129) lead to the famous Heisenberg uncertainty relation $\Delta x \Delta p_x \geq \hbar/2$.

PROBLEMS

3.1. Verify for the operators $\hat{A}$, $\hat{B}$, and $\hat{C}$ that
(*a*) $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$
(*b*) $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$

3.2. Using the $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$ states of a spin-$\frac{1}{2}$ particle as a basis, set up and solve as a problem in matrix mechanics the eigenvalue problem for $\hat{S}_n = \hat{\mathbf{S}} \cdot \mathbf{n}$, where the spin operator $\hat{\mathbf{S}} = \hat{S}_x\mathbf{i} + \hat{S}_y\mathbf{j} + \hat{S}_z\mathbf{k}$ and $\mathbf{n} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$. Show that the eigenstates may be written as

$$|+\mathbf{n}\rangle = \cos\frac{\theta}{2}|+\mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\mathbf{z}\rangle$$

$$|-\mathbf{n}\rangle = \sin\frac{\theta}{2}|+\mathbf{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\mathbf{z}\rangle$$

Rather than simply verifying that these are eigenstates by substituting into the eigenvalue equation, obtain these states by directly solving the eigenvalue problem, as in Section 3.6.

3.3. Show that the Pauli spin matrices satisfy $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}$, where i and j can take on the values 1, 2, and 3, with the understanding that $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$. Thus for $i = j$ show that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$, while for $i \neq j$ show that $\{\sigma_i, \sigma_j\} = 0$, where the curly brackets are called an *anticommutator*, which is defined by the relationship $\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$.

3.4. Verify that (*a*) $\boldsymbol{\sigma} \times \boldsymbol{\sigma} = 2i\boldsymbol{\sigma}$ and (*b*) $\boldsymbol{\sigma} \cdot \mathbf{a}\boldsymbol{\sigma} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}\mathbb{1} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b})$, where $\boldsymbol{\sigma} = \sigma_x\mathbf{i} + \sigma_y\mathbf{j} + \sigma_z\mathbf{k}$.

3.5. This problem demonstrates another way (also see Problem 3.2) to determine the eigenstates of $\hat{S}_n = \hat{\mathbf{S}} \cdot \mathbf{n}$. The operator

$$\hat{R}(\theta\mathbf{j}) = e^{-i\hat{S}_y\theta/\hbar}$$

rotates spin states by an angle θ counterclockwise about the y axis.
(*a*) Show that this rotation operator can be expressed in the form

$$\hat{R}(\theta\mathbf{j}) = \cos\frac{\theta}{2} - \frac{2i}{\hbar}\hat{S}_y\sin\frac{\theta}{2}$$

Suggestion: Use the states $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$ as a basis. Express the operator $\hat{R}(\theta\mathbf{j})$ in matrix form by expanding $\hat{R}$ in a Taylor series. Examine the explicit form for the matrices representing $\hat{S}_y^2$, $\hat{S}_y^3$, and so on.
(*b*) Apply $\hat{R}$ in matrix form to the state $|+\mathbf{z}\rangle$ to obtain the state $|+\mathbf{n}\rangle$ given in Problem 3.2 with $\phi = 0$, that is, rotated by angle θ in the x-z plane.

3.6. Derive (3.60).

3.7. Derive the Schwarz inequality

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

Suggestion: Use the fact that

$$\big(\langle\alpha| + \lambda^*\langle\beta|\big)\big(|\alpha\rangle + \lambda|\beta\rangle\big) \geq 0$$

and determine the value of λ that minimizes the left-hand side of the equation.

3.8. Show that the operator $\hat{C}$ defined through $[\hat{A}, \hat{B}] = i\hat{C}$ is Hermitian, provided the operators $\hat{A}$ and $\hat{B}$ are Hermitian.

3.9. Calculate ΔS_x and ΔS_y for an eigenstate of $\hat{S}_z$ for a spin-$\frac{1}{2}$ particle. Check to see if the uncertainty relation $\Delta S_x \Delta S_y \geq \hbar|\langle S_z\rangle|/2$ is satisfied. Repeat your calculation for an eigenstate of $\hat{S}_x$.

3.10. Use the matrix representations of the spin-$\frac{1}{2}$ angular momentum operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ in the S_z basis to verify explicitly through matrix multiplication that

$$\big[\hat{S}_x, \hat{S}_y\big] = i\hbar\hat{S}_z$$

3.11. Determine the matrix representations of the spin-$\frac{1}{2}$ angular momentum operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ using the eigenstates of $\hat{S}_y$ as a basis.

3.12. By examining their action on the basis states $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$, verify for a spin-$\frac{1}{2}$ particle that (*a*)

$$\hat{S}_z = (\hbar/2)|+\mathbf{z}\rangle\langle+\mathbf{z}| - (\hbar/2)|-\mathbf{z}\rangle\langle-\mathbf{z}|$$

and (*b*) the raising and lowering operators may be expressed as

$$\hat{S}_+ = \hbar|+\mathbf{z}\rangle\langle-\mathbf{z}| \qquad \text{and} \qquad \hat{S}_- = \hbar|-\mathbf{z}\rangle\langle+\mathbf{z}|$$

3.13. Repeat Problem 3.10 using the matrix representations (3.28) for a spin-1 particle in the J_z basis.

3.14. Use the spin-1 states $|1, 1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$ as a basis to form the matrix representations of the angular momentum operators and hence verify that the matrix representations (3.28) are correct.

3.15. Determine the eigenstates of $\hat{S}_x$ for a spin-1 particle in terms of the eigenstates $|1, 1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$ of $\hat{S}_z$.

3.16. A spin-1 particle exits an SG**z** device in a state with $S_z = \hbar$. The beam then enters an SG**x** device. What is the probability that the measurement of S_x yields the value 0?

3.17. A spin-1 particle is in the state

$$|\psi\rangle \xrightarrow[S_z \text{ basis}]{} \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}$$

(*a*) What are the probabilities that a measurement of S_z will yield the values $\hbar$, 0, or $-\hbar$ for this state? What is $\langle S_z\rangle$?

(*b*) What is $\langle S_x\rangle$ for this state? *Suggestion:* Use matrix mechanics to evaluate the expectation value.

(*c*) What is the probability that a measurement of S_x will yield the value $\hbar$ for this state?

3.18. Determine the eigenstates of $\hat{S}_n = \hat{\mathbf{S}} \cdot \mathbf{n}$ for a spin-1 particle, where the spin operator $\hat{\mathbf{S}} = \hat{S}_x\mathbf{i} + \hat{S}_y\mathbf{j} + \hat{S}_z\mathbf{k}$ and $\mathbf{n} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$. Use the matrix of this section. The eigenvalues of $\hat{J}_z$ are $\hbar$, 0, and $-\hbar$, which are the diagonal matrix elements of the matrix representing $\hat{J}_z$ in (3.28). The magnitude of the angular momentum for these states is given by $[1(1+1)]^{1/2}\hbar = (2)^{1/2}\hbar$.

4. There are four $j = \frac{3}{2}$ states: $|\frac{3}{2}, \frac{3}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$, and $|\frac{3}{2}, -\frac{3}{2}\rangle$. The magnitude of the angular momentum is $[\frac{3}{2}(\frac{3}{2}+1)]^{1/2}\hbar = (15)^{1/2}\hbar/2$.

As these examples illustrate, the magnitude $[j(j+1)]^{1/2}\hbar$ of the angular momentum is always bigger than the maximum projection $j\hbar$ on the z axis for any nonzero angular momentum. In Section 3.5 we will see how the uncertainty relations for angular momentum allow us to understand why the angular momentum does not line up along an axis.

3.4 THE MATRIX ELEMENTS OF THE RAISING AND LOWERING OPERATORS

We have seen in (3.40) and (3.42) that the action of the raising operator $\hat{J}_+$ on a state of angular momentum j is to create a state with the same magnitude of the angular momentum but with the z component increased by one unit of $\hbar$:

$$\hat{J}_+|j, m\rangle = c_+\hbar|j, m+1\rangle \tag{3.55}$$

while the action of the lowering operator is

$$\hat{J}_-|j, m\rangle = c_-\hbar|j, m-1\rangle \tag{3.56}$$

It is useful to determine the values of c_+ and c_-. Taking the inner product of the ket (3.55) with the corresponding bra and making use of (3.38), we obtain

$$\langle j, m|\hat{J}_-\hat{J}_+|j, m\rangle = c_+^*c_+\hbar^2\langle j, m+1|j, m+1\rangle \tag{3.57}$$

Substituting (3.47) for the operators, we find

$$\begin{aligned}\langle j, m|(\hat{\mathbf{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)|j, m\rangle &= [j(j+1) - m^2 - m]\hbar^2\langle j, m|j, m\rangle \\ &= c_+^*c_+\hbar^2\langle j, m+1|j, m+1\rangle\end{aligned} \tag{3.58}$$

Assuming the angular momentum states satisfy $\langle j, m|j, m\rangle = \langle j, m+1|j, m+1\rangle$, we can choose $c_+ = [j(j+1) - m(m+1)]^{1/2}$, or

$$\hat{J}_+|j, m\rangle = \sqrt{j(j+1) - m(m+1)}\,\hbar|j, m+1\rangle \tag{3.59}$$

Note that when $m = j$, the square root factor vanishes and the raising action terminates, as it must. Similarly, we can establish that

$$\hat{J}_-|j, m\rangle = \sqrt{j(j+1) - m(m-1)}\,\hbar|j, m-1\rangle \tag{3.60}$$

for which the square root factor vanishes when $m = -j$, as it must.

These results determine the matrix elements of the raising and lowering operators using the states $|j, m\rangle$ as a basis:

$$\begin{aligned}\langle j, m'|\hat{J}_+|j, m\rangle &= \sqrt{j(j+1) - m(m+1)}\,\hbar\langle j, m'|j, m+1\rangle \\ &= \sqrt{j(j+1) - m(m+1)}\,\hbar\delta_{m', m+1}\end{aligned} \tag{3.61}$$