3.6. Derive (3.60).

3.7. Derive the Schwarz inequality

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

Suggestion: Use the fact that

$$\big(\langle\alpha| + \lambda^*\langle\beta|\big)\big(|\alpha\rangle + \lambda|\beta\rangle\big) \geq 0$$

and determine the value of λ that minimizes the left-hand side of the equation.

3.8. Show that the operator $\hat{C}$ defined through $[\hat{A}, \hat{B}] = i\hat{C}$ is Hermitian, provided the operators $\hat{A}$ and $\hat{B}$ are Hermitian.

3.9. Calculate ΔS_x and ΔS_y for an eigenstate of $\hat{S}_z$ for a spin-$\frac{1}{2}$ particle. Check to see if the uncertainty relation $\Delta S_x \Delta S_y \geq \hbar|\langle S_z\rangle|/2$ is satisfied. Repeat your calculation for an eigenstate of $\hat{S}_x$.

3.10. Use the matrix representations of the spin-$\frac{1}{2}$ angular momentum operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ in the S_z basis to verify explicitly through matrix multiplication that

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$$

3.11. Determine the matrix representations of the spin-$\frac{1}{2}$ angular momentum operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ using the eigenstates of $\hat{S}_y$ as a basis.

3.12. By examining their action on the basis states $|+\mathbf{z}\rangle$ and $|-\mathbf{z}\rangle$, verify for a spin-$\frac{1}{2}$ particle that (*a*)

$$\hat{S}_z = (\hbar/2)|+\mathbf{z}\rangle\langle+\mathbf{z}| - (\hbar/2)|-\mathbf{z}\rangle\langle-\mathbf{z}|$$

and (*b*) the raising and lowering operators may be expressed as

$$\hat{S}_+ = \hbar|+\mathbf{z}\rangle\langle-\mathbf{z}| \qquad \text{and} \qquad \hat{S}_- = \hbar|-\mathbf{z}\rangle\langle+\mathbf{z}|$$

3.13. Repeat Problem 3.10 using the matrix representations (3.28) for a spin-1 particle in the J_z basis.

3.14. Use the spin-1 states $|1, 1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$ as a basis to form the matrix representations of the angular momentum operators and hence verify that the matrix representations (3.28) are correct.

3.15. Determine the eigenstates of $\hat{S}_x$ for a spin-1 particle in terms of the eigenstates $|1, 1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$ of $\hat{S}_z$.

3.16. A spin-1 particle exits an SG**z** device in a state with $S_z = \hbar$. The beam then enters an SG**x** device. What is the probability that the measurement of S_x yields the value 0?

3.17. A spin-1 particle is in the state

$$|\psi\rangle \underset{S_z \text{ basis}}{\longrightarrow} \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}$$

(*a*) What are the probabilities that a measurement of S_z will yield the values $\hbar$, 0, or $-\hbar$ for this state? What is $\langle S_z\rangle$?

(*b*) What is $\langle S_x\rangle$ for this state? *Suggestion:* Use matrix mechanics to evaluate the expectation value.

(*c*) What is the probability that a measurement of S_x will yield the value $\hbar$ for this state?

3.18. Determine the eigenstates of $\hat{S}_n = \hat{\mathbf{S}} \cdot \mathbf{n}$ for a spin-1 particle, where the spin operator $\hat{\mathbf{S}} = \hat{S}_x\mathbf{i} + \hat{S}_y\mathbf{j} + \hat{S}_z\mathbf{k}$ and $\mathbf{n} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$. Use the matrix representation of the rotation operator in Problem 3.19 to check your result when $\phi = 0$.

3.19. Find the state with $S_n = \hbar$ of a spin-1 particle, where $\mathbf{n} = \sin\theta\mathbf{i} + \cos\theta\mathbf{k}$, by rotating a state with $S_z = \hbar$ by angle θ counterclockwise about the y axis using the rotation operator $\hat{R}(\theta\mathbf{j}) = e^{-i\hat{S}_y\theta/\hbar}$. *Suggestion:* Use the matrix representation (3.104) for $\hat{S}_y$ in the S_z basis and expand the rotation operator in a Taylor series. Work out the matrices through the one representing $\hat{S}_y^3$ in order to see the pattern and show that

$$\hat{R}(\theta\mathbf{j}) \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \dfrac{1+\cos\theta}{2} & -\dfrac{\sin\theta}{\sqrt{2}} & \dfrac{1-\cos\theta}{2} \\ \dfrac{\sin\theta}{\sqrt{2}} & \cos\theta & -\dfrac{\sin\theta}{\sqrt{2}} \\ \dfrac{1-\cos\theta}{2} & \dfrac{\sin\theta}{\sqrt{2}} & \dfrac{1+\cos\theta}{2} \end{pmatrix}$$

3.20. A beam of spin-1 particles is sent through a series of three Stern-Gerlach measuring devices (Fig. 3.11). The first SG**z** device transmits particles with $S_z = \hbar$ and filters out particles with $S_z = 0$ and $S_z = -\hbar$. The second device, an SG**n** device, transmits particles with $S_n = \hbar$ and filters out particles with $S_n = 0$ and $S_n = -\hbar$, where the axis **n** makes an angle θ in the x-z plane with respect to the z axis. A last SG**z** device transmits particles with $S_z = -\hbar$ and filters out particles with $S_z = \hbar$ and $S_z = 0$.

(*a*) What fraction of the particles transmitted by the first SG**z** device will survive the third measurement? *Note:* The states with $S_n = \hbar$, $S_n = 0$, and $S_n = -\hbar$ in the S_z basis follow directly from applying the rotation operator given in Problem 3.19 to states with $S_z = \hbar$, $S_z = 0$, and $S_z = -\hbar$, respectively.

(*b*) How must the angle θ of the SG**n** device be oriented so as to maximize the number of particles that are transmitted by the final SG**z** device? What fraction of the particles survive the third measurement for this value of θ?

(*c*) What fraction of the particles survive the last measurement if the SG**n** device is removed from the experiment?

Repeat your calculation for parts (*a*), (*b*), and (*c*) if the last SG**z** device transmits particles with $S_z = 0$ only.

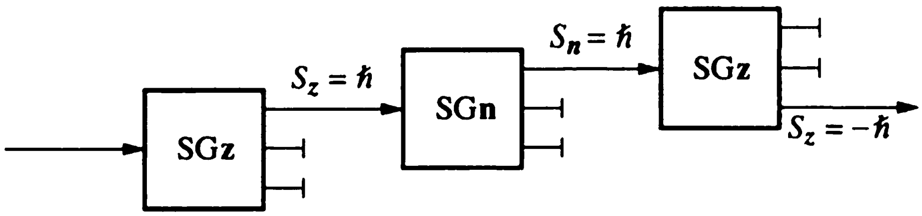

FIGURE 3.11

3.21. Introduce an angle θ defined by the relation $\cos\theta = J_z/|\mathbf{J}|$, reflecting the degree to which a particle's angular momentum lines up along the z axis. What is the smallest value of θ for (*a*) a spin-$\frac{1}{2}$ particle, (*b*) a spin-1 particle, and (*c*) a macroscopic spinning top?

3.22. Show that if the two Hermitian operators $\hat{A}$ and $\hat{B}$ have a *complete* set of eigenstates in common, the operators commute.

rotation operator $\hat{R}(d\phi\,\mathbf{n}) = 1 - i\hat{J}_n d\phi/\hbar$ for rotations by angle $d\phi$ about the axis specified by the unit vector $\mathbf{n}$ with the infinitesimal time translation operator (4.69). We can actually tie the rotation operator and the time-evolution operator together with a common thread—namely, symmetry. A symmetry operation is one that leaves the physical system unchanged or invariant. For example, if the Hamiltonian is invariant under rotations about an axis, the generator of rotations about that axis must commute with the Hamiltonian. But (4.77) then tells us that the component of the angular momentum along this axis is conserved, since its expectation value doesn't vary in time. Also, if the Hamiltonian is invariant under time translations, which simply means that $\hat{H}$ is independent of time, then of course energy is conserved. We will have more to say about symmetry, especially in Chapter 9, but this is our first indication of the important connection between symmetries of a physical system and conservation laws.

PROBLEMS

4.1. Show that unitarity of the infinitesimal time-evolution operator (4.4) requires that the Hamiltonian $\hat{H}$ be Hermitian.

4.2. Show that if the Hamiltonian depends on time *and* $[\hat{H}(t_1), \hat{H}(t_2)] = 0$, the time-development operator is given by

$$\hat{U}(t) = \exp\left[-\frac{i}{\hbar}\int_0^t dt'\,\hat{H}(t')\right]$$

4.3. Use (4.16) to verify that expectation value of an observable A does not change with time if the system is in an energy eigenstate (a stationary state) and $\hat{A}$ does not depend explicitly on time.

4.4. A beam of spin-$\frac{1}{2}$ particles with speed v_0 passes through a series of two SG**z** devices. The first SG**z** device transmits particles with $S_z = \hbar/2$ and filters out particles with $S_z = -\hbar/2$. The second SG**z** device transmits particles with $S_z = -\hbar/2$ and filters out particles with $S_z = \hbar/2$. Between the two devices is a region of length l_0 in which there is a uniform magnetic field B_0 pointing in the x direction. Determine the smallest value of l_0 such that exactly 25 percent of the particles transmitted by the first SG**z** device are transmitted by the second device. Express your result in terms of $\omega_0 = egB_0/2mc$ and v_0.

4.5. A beam of spin-$\frac{1}{2}$ particles in the $|+\mathbf{z}\rangle$ state enters a uniform magnetic field B_0 in the x-z plane oriented at an angle θ with respect to the z axis. At time T later, the particles enter an SG**y** device. What is the probability the particles will be found with $S_y = \hbar/2$? Check your result by evaluating the special cases $\theta = 0$ and $\theta = \pi/2$.

4.6. Verify that the expectation values (4.23), (4.28), and (4.30) for a spin-$\frac{1}{2}$ particle precessing in a uniform magnetic field B_0 in the z direction satisfy (4.16).

4.7. Use the data given in Fig. 4.3 to determine the g factor of the muon.

4.8. A spin-$\frac{1}{2}$ particle, initially in a state with $S_n = \hbar/2$ with $\mathbf{n} = \sin\theta\mathbf{i} + \cos\theta\mathbf{k}$, is in a constant magnetic field B_0 in the z direction. Determine the state of the particle at time t and determine how $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$ vary with time.

4.9. Derive Rabi's formula (4.45).

4.10. Express the Hamiltonian (4.57) for the ammonia molecule in the $|I\rangle$-$|II\rangle$ basis to obtain (4.61). Assume the electric field $\mathbf{E} = \mathbf{E}_0 \cos\omega t$. Compare this Hamiltonian

The state just picks up an overall phase as time progresses; thus, the physical state of the system does not change with time. We often call such an energy eigenstate a *stationary state* to emphasize this lack of time dependence.

You might worry that physics could turn out to be boring with a lot of emphasis on stationary states. However, if the initial state $|\psi(0)\rangle$ is a *superposition* of energy eigenstates with different energies, the *relative* phases between these energy eigenstates will change with time. Such a state is not a stationary state and the time-evolution operator will generate interesting time behavior. All we need to do to determine this time dependence is to express this initial state as a superposition of energy eigenstates, since we now know the action of the time-evolution operator on each of these states. We will see examples in Sections 4.3 and 4.5.

4.2 TIME DEPENDENCE OF EXPECTATION VALUES

The Schrödinger equation permits us to determine in general which variables exhibit time dependence for their expectation values. If we consider an observable A, then

$$\begin{aligned}
\frac{d}{dt}\langle A\rangle &= \frac{d}{dt}\langle\psi(t)|\hat{A}|\psi(t)\rangle \\
&= \left(\frac{d}{dt}\langle\psi(t)|\right)\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\left(\frac{d}{dt}|\psi(t)\rangle\right) + \langle\psi(t)|\frac{\partial\hat{A}}{\partial t}|\psi(t)\rangle \\
&= \left(\frac{1}{-i\hbar}\langle\psi(t)|\hat{H}\right)\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\left(\frac{1}{i\hbar}\hat{H}|\psi(t)\rangle\right) + \langle\psi(t)|\frac{\partial\hat{A}}{\partial t}|\psi(t)\rangle \\
&= \frac{i}{\hbar}\langle\psi(t)|\left[\hat{H},\hat{A}\right]|\psi(t)\rangle + \langle\psi(t)|\frac{\partial\hat{A}}{\partial t}|\psi(t)\rangle
\end{aligned} \tag{4.16}$$

The appearance of the last term involving $\partial\hat{A}/\partial t$ in this equation allows for the possibility that the operator depends *explicitly* on time. Equation (4.16) shows that provided the operator corresponding to a variable does not have any explicit time dependence ($\partial\hat{A}/\partial t = 0$), the expectation value of that variable will be a constant of the motion whenever the operator commutes with the Hamiltonian.

What do we mean by explicit time dependence in the operator? Our examples in Sections 4.3 and 4.4 will probably illustrate this best. The Hamiltonian for a spin-$\frac{1}{2}$ particle in a constant magnetic field is given in (4.17). There is no explicit t dependence in $\hat{H}$; therefore substituting $\hat{H}$ for the operator $\hat{A}$ in (4.16) indicates that energy is conserved, since $\hat{H}$ of course commutes with itself. However, if we examine the Hamiltonian (4.34) for a spin-$\frac{1}{2}$ particle in a time-dependent magnetic field, we see explicit time dependence within the Hamiltonian in the factor $\cos\omega t$. Such a Hamiltonian does not lead to an expectation value for the energy of the spin system that is independent of time because $\partial\hat{H}/\partial t \neq 0$. There is clearly