

Thus the position-space and momentum-space wave functions form a Fourier transform pair.

PROBLEMS

- 6.1. (a) Use induction to show that $[\hat{x}^n, \hat{p}_x] = i\hbar n \hat{x}^{n-1}$.
 (b) By expanding $F(x)$ in a Taylor series, show that

$$[F(\hat{x}), \hat{p}_x] = i\hbar \frac{\partial F}{\partial x}(\hat{x})$$

- (c) For the one-dimensional Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x})$$

show that

$$\frac{d\langle p_x \rangle}{dt} = \left\langle -\frac{dV}{dx} \right\rangle$$

- 6.2. Show that

$$\langle p | \hat{x} | \psi \rangle = i\hbar \frac{\partial}{\partial p} \langle p | \psi \rangle$$

and

$$\langle \varphi | \hat{x} | \psi \rangle = \int dp \langle p | \varphi \rangle^* i\hbar \frac{\partial}{\partial p} \langle p | \psi \rangle$$

What do these results suggest for how you should represent the position operator in momentum space?

- 6.3. Show for infinitesimal translations $|\psi\rangle \rightarrow |\psi'\rangle = \hat{T}(\delta x)|\psi\rangle$ that $\langle x \rangle \rightarrow \langle x \rangle + \delta x$ and $\langle p_x \rangle \rightarrow \langle p_x \rangle$.
 6.4. (a) Show for a free particle of mass m initially in the state

$$\psi(x) = \langle x | \psi \rangle = \frac{1}{\sqrt{\sqrt{\pi}a}} e^{-x^2/2a^2}$$

that

$$\psi(x, t) = \langle x | \psi(t) \rangle = \frac{1}{\sqrt{\sqrt{\pi}[a + (i\hbar t/ma^2)]}} e^{-x^2/2a^2[1 + (i\hbar t/ma^2)]}$$

and therefore

$$\Delta x = \frac{a}{\sqrt{2}} \sqrt{1 + \left(\frac{\hbar t}{ma^2}\right)^2}$$

- (b) What is Δp_x at time t ? *Suggestion:* Use the momentum-space wave function to evaluate Δp_x .

6.5. Consider a wave packet defined by

$$\langle p|\psi\rangle = \begin{cases} 0 & p < -P/2 \\ N & -P/2 < p < P/2 \\ 0 & p > P/2 \end{cases}$$

- (a) Determine a value for N such that $\langle\psi|\psi\rangle = 1$ using the momentum-space wave function directly.
 (b) Determine $\langle x|\psi\rangle = \psi(x)$.
 (c) Sketch $\langle p|\psi\rangle$ and $\langle x|\psi\rangle$. Use reasonable estimates of Δp_x from the form of $\langle p|\psi\rangle$ and Δx from the form of $\langle x|\psi\rangle$ to estimate the product $\Delta x \Delta p_x$. Check that your result is independent of the value of P . *Note:* Simply estimate rather than actually calculate the uncertainties.
- 6.6. (a) Show that $\langle p_x\rangle = 0$ for a state with a *real* wave function $\langle x|\psi\rangle$.
 (b) Show that if the wave function $\langle x|\psi\rangle$ is modified by a position-dependent phase

$$\langle x|\psi\rangle \rightarrow e^{ip_0 x/\hbar} \langle x|\psi\rangle$$

then

$$\langle x\rangle \rightarrow \langle x\rangle \quad \text{and} \quad \langle p_x\rangle \rightarrow \langle p_x\rangle + p_0$$

- 6.7. Establish that the position operator \hat{x} is Hermitian (a) by showing that $\langle\phi|\hat{x}|\psi\rangle = \langle\psi|\hat{x}|\phi\rangle^*$ or (b) by taking the adjoint of the position-momentum commutation relation (6.31).
 6.8. Without using exact mathematics—that is, using only arguments of curvature, symmetry, and semiquantitative estimates of wavelength—sketch the energy eigenfunctions for the ground state, first-excited state, and second-excited state of a particle in the potential energy well $V(x) = a|x|$ shown in Fig. 6.15. This potential energy has been suggested as arising from the force exerted by one quark on another quark.

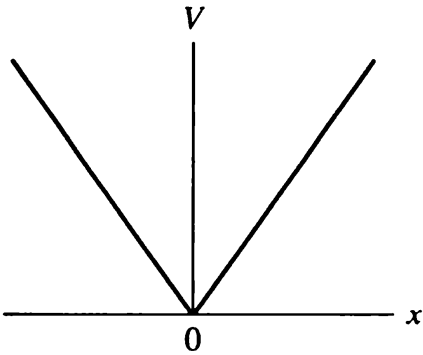


FIGURE 6.15

- 6.9. The one-dimensional time-independent Schrödinger equation for the potential energy discussed in Problem 6.8 is

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - a|x|)\psi = 0$$

Define $E = \varepsilon(\hbar^2 a^2/m)^{1/3}$ and $x = z(\hbar^2/ma)^{1/3}$.

- (a) Show that ε and z are dimensionless.
 (b) Show that the Schrödinger equation can be expressed in the form