matrix representations that the position-momentum commutation relation (7.7) is satisfied.

7.3. Show that properly normalized eigenstates of the harmonic oscillator are given by (7.38). *Suggestion:* Use induction.

7.4. Use $\hat{a}|0\rangle = 0$ and therefore $\langle p|\hat{a}|0\rangle = 0$ to solve directly for $\langle p|0\rangle$, the ground-state wave function of the harmonic oscillator in momentum space. Normalize the wave function. *Hint:* Recall the result of Problem 6.2,

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$$

7.5. Derive (7.56) and (7.57).

7.6. A particle of mass m in the one-dimensional harmonic oscillator is in a state for which a measurement of the energy yields the values $\hbar\omega/2$ or $3\hbar\omega/2$, each with a probability of one-half. The average value of the momentum $\langle p_x\rangle$ at time $t = 0$ is $(m\omega\hbar/2)^{1/2}$. This information specifies the state of the particle completely. What is this state and what is $\langle p_x\rangle$ at time t?

7.7. (*a*) Determine the size of the classical turning point x_0 for a harmonic oscillator in its ground state with a mass of 1000 kg and a frequency of 1000 Hz. Compare your result with the size of a proton. A bar of aluminum of roughly this mass and tuned to roughly this frequency (called a Weber bar) is used in attempts to detect gravity waves.

(*b*) Suppose that the bar absorbs energy in the form of a graviton and makes a transition from a state with energy E_n to a state with energy E_{n+1}. Show that the change in length of such a bar is given approximately by $x_0(2/n)^{1/2}$ for large n.

(*c*) To what n, on the average, is the oscillator excited by thermal energy if the bar is cooled to 1 K?

7.8. Show that in the superposition of adjacent energy states (7.63) the average value of the position of the particle is given by

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle = A\cos(\omega t + \delta)$$

and the average value of the momentum is given by

$$\langle p_x\rangle = \langle\psi|\hat{p}_x|\psi\rangle = -m\omega A\sin(\omega t + \delta)$$

in accord with Ehrenfest's theorem, (6.33) and (6.34).

7.9. A small cylindrical tube is drilled through the earth, passing through the center. A mass m is released essentially at rest at the surface. Assuming the density of the earth is uniform, show that the mass executes simple harmonic motion and determine the frequency ω. Determine the approximate quantum number n for this state of the mass, using a typical macroscopic value for the magnitude of the mass m. Explain why a *single* quantum number n is inadequate to specify the state.

7.10. Prove that the parity operator $\hat{\Pi}$ is Hermitian.

7.11. Substitute $\psi(x) = Ne^{-ax^2}$ into the position-space energy eigenvalue equation (7.66) and determine the value of the constant a that makes this function an eigenfunction. What is the corresponding energy eigenvalue?

7.12. Calculate the probability that a particle in the ground state of the harmonic oscillator is located in a classically disallowed region, namely, where $V(x) > E$. Obtain a numerical value for the probability. *Suggestion:* Express your integral in terms of a dimensionless variable and compare with the tabulated values of the error function.

work because as $\Delta x \to 0$, $\Delta p_x \to \infty$ in order to satisfy $\Delta x \Delta p_x \geq \hbar/2$. Similarly, trying to put the particle in a state with zero momentum to minimize the kinetic energy implies $\Delta p_x \to 0$, which forces $\Delta x \to \infty$. Thus nature must choose a tradeoff in which the particle has both nonzero Δx and Δp_x and, therefore, nonzero energy. Explicitly, for the ground state

$$\begin{aligned}
\Delta x^2 &= \frac{\hbar}{2m\omega}\langle 0|(\hat{a} + \hat{a}^\dagger)^2|0\rangle \\
&= \frac{\hbar}{2m\omega}\langle 0|[\hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}]|0\rangle \\
&= \frac{\hbar}{2m\omega}\langle 0|\hat{a}\hat{a}^\dagger|0\rangle = \frac{\hbar}{2m\omega}\langle 1|1\rangle = \frac{\hbar}{2m\omega}
\end{aligned} \tag{7.54}$$

and

$$\begin{aligned}
\Delta p_x^2 &= -\frac{m\omega\hbar}{2}\langle 0|(\hat{a} - \hat{a}^\dagger)^2|0\rangle \\
&= -\frac{m\omega\hbar}{2}\langle 0|[\hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}]|0\rangle \\
&= \frac{m\omega\hbar}{2}\langle 0|\hat{a}\hat{a}^\dagger|0\rangle = \frac{m\omega\hbar}{2}\langle 1|1\rangle = \frac{m\omega\hbar}{2}
\end{aligned} \tag{7.55}$$

Notice that $\Delta x \Delta p_x = \hbar/2$ for the ground state. That the ground state is a minimum uncertainty state was already apparent from the Gaussian form of the ground-state wave function (7.44), given the discussion in Section 6.6. For the excited states, we can establish in a similar fashion that

$$\Delta x = \sqrt{\left(n + \tfrac{1}{2}\right)\frac{\hbar}{m\omega}} \tag{7.56}$$

$$\Delta p_x = \sqrt{\left(n + \tfrac{1}{2}\right)m\omega\hbar} \tag{7.57}$$

and

$$\Delta x \Delta p_x = \left(n + \tfrac{1}{2}\right)\hbar \tag{7.58}$$

A good illustration of the effects of this zero-point energy is the unusual behavior of helium. Helium is the only substance that does not solidify at sufficiently low temperatures at atmospheric pressure. Rather, it is necessary to apply a pressure of at least 25 atmospheres. For substances other than helium, the uncertainty in the position of the nuclei in the ground state is in general quite small compared to the spacing between the nuclei, which is why these substances solidify at atmospheric pressure at sufficiently low temperature. In fact, increasing the temperature populates the higher vibrational states and increases the uncertainty, as (7.56) indicates. These substances melt when the uncertainty becomes comparable to the spacing between the nuclei in the solid. For helium, even in

Time dependence for the harmonic oscillator results from the system being in a superposition of energy eigenstates with different energies. If we assume the initial state is a superposition of two *adjacent* energy states,

$$|\psi(0)\rangle = c_n|n\rangle + c_{n+1}|n+1\rangle \tag{7.62}$$

then

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\hat{H}t/\hbar}|\psi(0)\rangle \\ &= e^{-i(n+1/2)\omega t}\left(c_n|n\rangle + c_{n+1}e^{-i\omega t}|n+1\rangle\right) \end{aligned} \tag{7.63}$$

In particular, we can take advantage of the expression (7.11) for the position operator in terms of the raising and lowering operators to evaluate the expectation value of the position of the particle and show that it behaves as one would expect for a classical particle, namely,

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle = A\cos(\omega t + \delta) \tag{7.64}$$

See Problem 7.8.

7.9 SOLVING THE SCHRÖDINGER EQUATION IN POSITION SPACE

There is another technique for determining the energy eigenvalues and the position-space eigenfunctions of the harmonic oscillator that we will find particularly useful when we solve the three-dimensional Schrödinger equation in Chapter 10. Rather than take advantage of the operator techniques of Section 7.2, we solve the energy eigenvalue equation

$$\langle x|\hat{H}|E\rangle = \langle x|\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)|E\rangle = E\langle x|E\rangle \tag{7.65}$$

directly in position space, as in Chapter 6. Using the results of that chapter, we can express this equation as

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\langle x|E\rangle + \frac{1}{2}m\omega^2x^2\langle x|E\rangle = E\langle x|E\rangle \tag{7.66}$$

The position-space energy eigenvalue equation (7.66) is a nontrivial second-order differential equation. To make its structure a little more apparent, it is good to introduce the dimensionless variable

$$y = \sqrt{\frac{m\omega}{\hbar}}\,x \tag{7.67}$$

where the factor $\sqrt{m\omega/\hbar}$ is a factor with the dimensions of inverse length that occurs naturally in the problem. We call the wave function

$$\langle x|E\rangle = \psi(y) \tag{7.68}$$

where in the next to last step we have kept just the leading order in the infinitesimal δx.[4] Comparing (6.29) and (6.30), we see that the position operator and the generator of translations obey the commutation relation

$$\left[\hat{x}, \hat{p}_x\right] = i\hbar \tag{6.31}$$

Given the pivotal role that the commutation relations (3.14) played in our discussion of angular momentum, it is probably not surprising to find that the commutation relation (6.31) plays a very important role in our discussion of wave mechanics.

In order to ascertain the physical significance of the generator of translations, we next examine the time evolution of a particle of mass m moving in one dimension. Continuing to neglect the spin degrees of freedom of the particle,[5] we can write the Hamiltonian as

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x}) \tag{6.32}$$

where we have expressed the kinetic energy of the particle in terms of the momentum and added a potential energy term V. Note that we are denoting the momentum operator by the same symbol as the generator of translations. We will now show that for quantum mechanics to yield predictions about the time evolution that are in accord with classical physics when appropriate, it is necessary that the momentum operator satisfy the commutation relation (6.31). Using (4.16), we can calculate the time rate of change of the expectation value of the position of the particle:

$$\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{i}{\hbar}\langle\psi|[\hat{H}, \hat{x}]|\psi\rangle = \frac{i}{\hbar}\langle\psi|\left[\frac{\hat{p}_x^2}{2m}, \hat{x}\right]|\psi\rangle \\
&= \frac{i}{2m\hbar}\langle\psi|\left(\hat{p}_x[\hat{p}_x, \hat{x}] + [\hat{p}_x, \hat{x}]\hat{p}_x\right)|\psi\rangle \\
&= \frac{\langle\psi|\hat{p}_x|\psi\rangle}{m} = \frac{\langle p_x\rangle}{m}
\end{aligned} \tag{6.33}$$

Moreover, you may also check (see Problem 6.1) that

$$\begin{aligned}
\frac{d\langle p_x\rangle}{dt} &= \frac{i}{\hbar}\langle\psi|[\hat{H}, \hat{p}_x]|\psi\rangle \\
&= \left\langle -\frac{dV}{dx}\right\rangle
\end{aligned} \tag{6.34}$$

[4] If this step bothers you, see also the discussion in going from (6.38) to (6.39). Here too, we can shift the integration variable ($x' = x + \delta x$), expand the wave function in a Taylor series, and retain only the leading-order term.

[5] It's hard to worry much about angular momentum in a one-dimensional world.