EXAMEN DEPARTAMENTAL GLOBAL DE CÁLCULO DIFERENCIAL E INTEGRAL II,

Horario: ____________________, **Grupo**: ________

Trimestre 10-I. Abril 8 de 2011.

Alumno (Nombre y matrícula):

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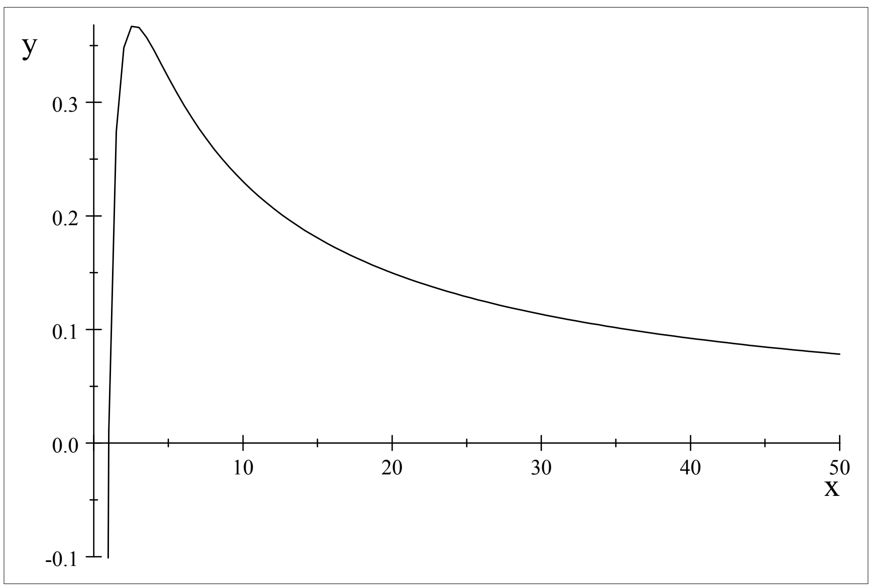
PARTE I.

(**1**) [**10**]. Obtener la gráfica de $f(x) = \frac{\ln(x)}{x}$ determinando: Dominio y raices; los intervalos de monotonia, los máximos y minimos relativos; los intervalos de concavidad y los puntos de inflexión.

RESPUESTA:

$f(x) = \frac{\ln(x)}{x}$



Dominio: $\mathbf{R}^+$

Raíz: $\frac{\ln(x_1)}{x_1} = 0; \ln(x_1) = 0; x_1 = 1.$

Intervalos de monotonia.

$\frac{d}{dx}\frac{\ln(x)}{x} = -\frac{1}{x^2}(\ln x - 1).$

Es monotona creciente si $x < e,$ ya que $-\frac{1}{x^2}(\ln x - 1) > 0;$ cuando $(\ln x - 1) < 0; \ln x < 1; x < e.$

Es monotona decreciente si $e < x,$ ya que $-\frac{1}{x^2}(\ln x - 1) < 0;$ cuando $(\ln x - 1) > 0; \ln x > 1; x > e.$

Su punto critico es $-\frac{1}{x_2^2}(\ln x_2 - 1) = 0 \Rightarrow \ln x_2 = 1 \Rightarrow x_2 = e$ donde $f(e) = \frac{1}{e} \approx 0.36788$

Es un máximo porque $\frac{d^2}{dx^2}f(x_2) < 0.$

Se tiene $\frac{d^2}{dx^2}\frac{\ln(x)}{x} = \frac{1}{x^3}(2\ln x - 3)$ en $x_2 = e$ es $\frac{1}{e^3}(2\ln e - 3) = -\frac{1}{e^3} < 0.$

Puntos de inflexión. Se obtienen de $\frac{d^2}{dx^2}f(x) = 0$.

$\frac{d^2}{dx^2}f(x_3) = \frac{1}{x_3^3}(2\ln x_3 - 3) = 0 \Rightarrow \ln x_3 = \frac{3}{2} \Rightarrow x_3 = e^{\frac{3}{2}} \approx 4.4817$

Además $\frac{d^3}{dx^3}f(x) = -\frac{1}{x^4}(6\ln x - 11)$ que es no cero en x_3, ya que

$-\frac{1}{\left(e^{\frac{3}{2}}\right)^4}\left(6\ln e^{\frac{3}{2}} - 11\right) = \frac{2}{e^6} \approx 4.9575 \times 10^{-3}$.

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(2) [5] Obtener la derivada de la función

$$h(x) = \arccos^2(1 + \ln(x)) + xe^{-2x}$$

RESPUESTA.

$\frac{d}{dx}h(x) = \frac{d}{dx}(\arccos^2(1 + \ln(x)) + xe^{-2x}) = \frac{d}{dx}\arccos^2(1 + \ln(x)) + \frac{d}{dx}xe^{-2x}$

Calculamos en forma seperada:

(1) $\frac{d}{dx}\arccos^2(1 + \ln(x)) = 2\arccos(1 + \ln(x))\frac{d}{dx}\arccos(1 + \ln(x)) =$

$2\arccos(1 + \ln(x))\frac{-\frac{d}{dx}(1+\ln(x))}{\sqrt{1-(1+\ln(x))^2}} = -\frac{2}{x}\frac{\arccos(1+\ln(x))}{\sqrt{1-(1+\ln(x))^2}} = -\frac{2}{x}\frac{\arccos(\ln x+1)}{\sqrt{-(\ln x)(\ln x+2)}}$

(2) $\frac{d}{dx}xe^{-2x} = -e^{-2x}(2x - 1)$

Se tiene $\frac{d}{dx}h(x) = -\frac{2}{x}\frac{\arccos(\ln x+1)}{\sqrt{-(\ln x)(\ln x+2)}} - e^{-2x}(2x - 1)$.

(**3**) [**25**]. Calcular las integrales:

a) $\int \frac{-7}{x}\sec^2(\ln(x))dx$

b) $\int_0^4 \frac{x}{3+\sqrt{x}}dx$

RESPUESTA.

a) $\int \frac{-7}{x}\sec^2(\ln(x))dx = -7\int \frac{1}{x}\sec^2(\ln(x))dx$, con la sustitución $du = \frac{1}{x}dx; u = \ln(x)$.

$-7\int \sec^2(u)du = -7\tan u + C$.

Por tanto $\int \frac{-7}{x}\sec^2(\ln(x))dx = -7\tan(\ln(x)) + C$

b) $\int_0^4 \frac{x}{3+\sqrt{x}}dx$

$\int \frac{x}{3+\sqrt{x}}dx$ con la sustitución $u^2 = x; 2udu = dx$;

$2\int \frac{u^3}{3+u}du = 2\int\left(u^2 - 3u + 9 - \frac{27}{3+u}\right)du =$

$2\int u^2du - 6\int udu + 18\int du - 54\int \frac{1}{3+u}du =$

$\frac{2}{3}u^3 - \frac{6}{2}u^2 + 18u - 54\ln(3 + u) = \frac{2}{3}x^{\frac{3}{2}} - \frac{6}{2}x + 18x^{\frac{1}{2}} - 54\ln\left(3 + x^{\frac{1}{2}}\right) + C$.

Comprobación

$\frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{6}{2}x + 18x^{\frac{1}{2}} - 54\ln\left(3 + x^{\frac{1}{2}}\right)\right) = \frac{x}{\sqrt{x}+3}$

Por tanto:

$\int_0^4 \frac{x}{3+\sqrt{x}}dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{6}{2}x + 18x^{\frac{1}{2}} - 54\ln\left(3 + x^{\frac{1}{2}}\right)\right]_0^4 =$

$\frac{2}{3}\left(4^{\frac{3}{2}}\right) - \frac{6}{2}4 + 18\left(4^{\frac{1}{2}}\right) - 54\ln\left(3 + 4^{\frac{1}{2}}\right) - (-54\ln(3)) =$

$\frac{2}{3}(2^3) - 12 + 36 - 54\ln(5) + 54\ln(3) =$

$\frac{16}{3} - 12 + 36 + 54(\ln(3) - \ln(5)) = 1.7487$

(**4**) Calcular $y'(\frac{\pi}{4})$ para $y(x) = \int_{\frac{1}{3}}^{\arccos(x)} \frac{\cos(t)}{t}dt$

RESPUESTA.

$y(x) = \int_{\frac{1}{3}}^{\arccos(x)} \frac{\cos(t)}{t}dt$

$y'(x) = \frac{d}{dx}\int_{\frac{1}{3}}^{\arccos(x)} \frac{\cos(t)}{t}dt = \frac{\cos(\arccos(x))}{\arccos(x)}\frac{d}{dx}\arccos(x) = -\frac{\cos(\arccos(x))}{\arccos(x)}\frac{1}{\sqrt{1-x^2}} = -\frac{1}{\arccos(x)}\frac{x}{\sqrt{1-x^2}}$

$y'(\frac{\pi}{4}) = -\frac{1}{\arccos\left(\frac{\pi}{4}\right)}\frac{\frac{\pi}{4}}{\sqrt{1-\left(\frac{\pi}{4}\right)^2}} = -\frac{1}{\frac{\sqrt{2}}{2}}\frac{\frac{\pi}{4}}{\sqrt{1-\left(\frac{\pi}{4}\right)^2}} = -\frac{1}{4}\sqrt{2}\frac{\pi}{\sqrt{1-\frac{1}{16}\pi^2}} \approx -1.7944$

PARTE II.

(**5**) Resolver

a) [10] $\int_0^{\frac{\pi}{6}} e^{-3x}\cos(3x)dx$

b) [10] $\int x\arcsin(3x)dx$

c) $\int \tan^3(x)\sec^3(x)dx$

RESPUESTA.

a) [10] $\int_0^{\frac{\pi}{6}} e^{-3x}\cos(3x)dx = 0.20131$

$\int e^{-3x}\cos(3x)dx = -\frac{1}{6}e^{-3x}(\cos 3x - \sin 3x)$

$\int_0^{\frac{\pi}{6}} e^{-3x}\cos(3x)dx = \left[-\frac{1}{6}e^{-3x}(\cos 3x - \sin 3x)\right]_0^{\frac{\pi}{6}} =$

$-\frac{1}{6}e^{-3\frac{\pi}{6}}\left(\cos 3\frac{\pi}{6} - \sin 3\frac{\pi}{6}\right) - \left(-\frac{1}{6}\right) =$

$-\frac{1}{6}e^{-\frac{\pi}{2}}(\cos\frac{\pi}{2} - \sin\frac{\pi}{2}) + \frac{1}{6} =$

$-\frac{1}{6}e^{-\frac{\pi}{2}}(-1) + \frac{1}{6} = \frac{1}{6}(1 + e^{-\frac{\pi}{2}}) = 0.20131$

b) [10] $\int x\arcsin(3x)dx = \frac{1}{2}x^2\arcsin 3x - \frac{1}{36}\arcsin 3x + \frac{1}{12}x\sqrt{1-9x^2}$

Comprobación.

$\frac{d}{dx}\left(\frac{1}{2}x^2\arcsin 3x - \frac{1}{36}\arcsin 3x + \frac{1}{12}x\sqrt{1-9x^2}\right) = x\arcsin 3x$

otra

$\frac{d}{dx}\left(\frac{1}{2}x^2\arcsin 3x + \frac{3}{2}\sqrt{1-C}\right) = \frac{1}{(1-9x^2)^{\frac{3}{2}}}(27x^2-2)$

c) $\int \tan^3(x)\sec^3(x)dx = \int \tan(x)\tan^2(x)\sec^3(x)dx = \int \tan(x)(\sec^2(x)-1)\sec^3(x)dx =$

$\int \tan(x)\sec^5(x)dx - \int \tan(x)\sec^3(x)dx = \int \sec^4(x)(\sec(x)\tan(x)dx) - \int \sec^2(x)(\sec(x)\tan(x)$

con el cambio de varaiable $u = \sec(x); du = \sec(x)\tan(x)dx$ se tiene

$\int u^4 du - \int u^2 du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C =$

$\frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C.$

Comprobación.

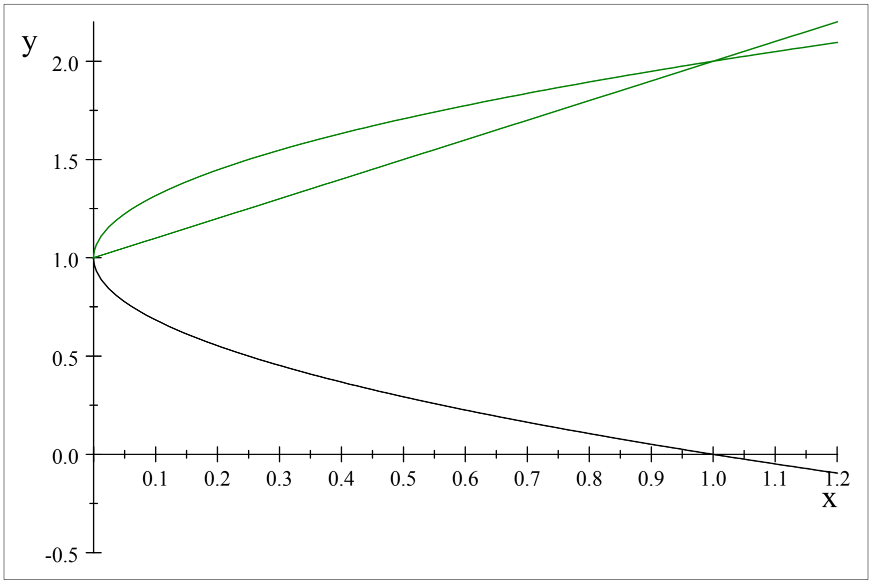
$\frac{d}{dx}\left(\frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x)\right) = \left(\frac{5}{5}\sec^4(x)\sec(x)\tan(x) - \frac{3}{3}\sec^2(x)\sec(x)\tan(x)\right) =$

$\sec^5(x)\tan(x) - \sec^3(x)\tan(x) = \sec^3(x)\tan(x)(\sec^2(x)-1) = \sec^3(x)\tan^3(x)$

(**6**) [**10**]. Calcular el área encerrada por las curvas $(y-1)^2 = x, \quad x - y + 1 = 0.$

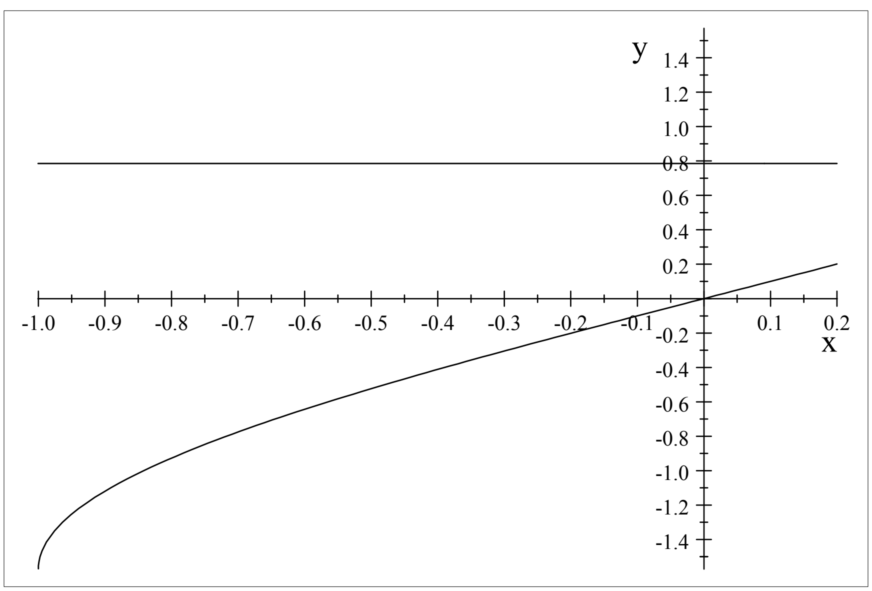
RESPUESTA

La grafica de las curvas es:



Las intersecciones ocurren en $x = 0$ y $x = 1$.
Las funciones son $f(x) = \sqrt{x} + 1$, $g(x) = x + 1$.
La integral de área es $\int_0^1 (f(x) - g(x))dx = \int_0^1 (\sqrt{x} + 1 - (x+1))dx = \int_0^1 (\sqrt{x} - x)dx =$
$\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \approx 0.16667$

(**7**) [**10**].Calcular el volumen del sólido de revolución obtenido al rotar, alrededor de la recta $x = -1$, la region limitada por la gráfica de $f(x) = \arcsin(x)$, la recta $y = \frac{\pi}{4}$ y el eje Y.
RESPUESTA.
$\arcsin(x)$



El radio del volumen de revolución está dado por

$$R(y) = \begin{cases} 1 - \sin(y) & y \in [-\frac{\pi}{2}, 0] \\ 1 & y \in [0, \frac{\pi}{4}] \end{cases}$$

$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \pi[R(y)]^2 dy = \int_{-\frac{\pi}{2}}^{0} \pi[1-\sin(y)]^2 dy + \int_{0}^{\frac{\pi}{4}} \pi[1]^2 dy =$

$\int \pi[1-\sin(y)]^2 dy = \frac{3}{2}\pi y - \frac{1}{4}\pi \sin 2y + 2\pi \cos y$

$[\frac{3}{2}\pi y - \frac{1}{4}\pi \sin 2y + 2\pi \cos y]_{\frac{-\pi}{2}}^{0} + [\pi y]_0^{\frac{\pi}{4}} =$

$[\frac{3}{2}\pi 0 - \frac{1}{4}\pi \sin 20 + 2\pi \cos 0] - [\frac{3}{2}\pi \frac{-\pi}{2} - \frac{1}{4}\pi \sin 2\frac{-\pi}{2} + 2\pi \cos \frac{-\pi}{2}] + \pi \frac{\pi}{4} =$

$2\pi + \frac{3}{4}\pi^2 + \frac{1}{4}\pi^2 = 2\pi + \pi^2 \approx 16.153$

PARTE III.

(8) Resolver

a) [**15**] $\int \frac{x^2}{(1-9x^2)^{\frac{3}{2}}} dx$

b) [**15**] $\int \frac{4x^3-3x^2+72x-108}{x^4+36x^2} dx$

c) [**5**] $\int_1^{\infty} \frac{\ln(x)}{x} dx$

RESPUESTA.

a) [**15**] $\int \frac{x^2}{(1-9x^2)^{\frac{3}{2}}} dx = \frac{1}{9}\int \frac{9x^2}{(1-9x^2)^{\frac{3}{2}}} dx$, con $u = 3x; du = 3dx$;

se tiene $\frac{1}{9}\int \frac{9x^2}{(1-9x^2)^{\frac{3}{2}}} dx = \frac{1}{27}\int \frac{u^2}{(1-u^2)^{\frac{3}{2}}} du$;

con una sustitución trigonometrica $u = \sin(\theta); du = \cos(\theta)d\theta$;

$\frac{1}{27}\int \frac{\sin^2(\theta)}{(1-\sin^2(\theta))^{\frac{3}{2}}} \cos(\theta)d\theta = \frac{1}{27}\int \frac{\sin^2(\theta)}{(\cos^2(\theta))^{\frac{3}{2}}} \cos(\theta)d\theta = \frac{1}{27}\int \frac{\sin^2(\theta)}{\cos^3(\theta)} \cos(\theta)d\theta =$

$\frac{1}{27}\int \frac{\sin^2(\theta)}{\cos^2(\theta)} d\theta = \frac{1}{27}\int \tan^2(\theta)d\theta = \frac{1}{27}\int(\sec^2(\theta)-1)d\theta = \frac{1}{27}\int \sec^2(\theta)d\theta - \frac{1}{27}\int d\theta =$

$\frac{1}{27}\tan(\theta) - \frac{1}{27}\theta + C = \frac{1}{27}\frac{u}{\sqrt{1-u^2}} - \arcsin(u) + C = \frac{1}{27}\frac{3x}{\sqrt{1-9x^2}} - \frac{1}{27}\arcsin(3x) + C$

Comprobación

$\frac{d}{dx}\left(\frac{1}{27}\frac{3x}{\sqrt{1-9x^2}} - \frac{1}{27}\arcsin(3x) + C\right) = \frac{x^2}{(1-9x^2)^{\frac{3}{2}}}$

b) [**15**] $\int \frac{4x^3-3x^2+72x-108}{x^4+36x^2} dx = \int \frac{4x^3-3x^2+72x-108}{x^2(x^2+36)} dx$

Por fracciones parciales

$\frac{4x^3-3x^2+72x-108}{x^2(x^2+36)} = \frac{ax+b}{x^2} + \frac{cx+d}{x^2+36}$

$4x^3 - 3x^2 + 72x - 108 = (ax+b)(x^2+36) + (cx+d)x^2 = (ax+b)(x^2+36) + (cx+d)x^2 =$
$36b + ax^3 + bx^2 + cx^3 + dx^2 + 36ax = x^3(a+c) + x^2(b+d) + x(36a) + 36b$

$4x^3 - 3x^2 + 72x - 108 = x^3(a+c) + x^2(b+d) + x(36a) + 36b$

Se tiene

$4 = a+c; -3 = b+d; 72 = 36a; -108 = 36b$

Por tanto $b = -3; a = 2; d = -3+3 = 0; c = 4-a = 2$

Se tiene $a = 2; b = -3; c = 2$ y $d = 0$.

Es decir $\frac{4x^3-3x^2+72x-108}{x^2(x^2+36)} = \frac{2x-3}{x^2} + \frac{2x}{x^2+36}$

$\int \frac{4x^3-3x^2+72x-108}{x^2(x^2+36)} dx = \int\left(\frac{2x-3}{x^2} + \frac{2x}{x^2+36}\right)dx = \int \frac{2x-3}{x^2} dx + \int \frac{2x}{x^2+36} dx =$

$\frac{1}{x}(2x\ln x + 3) + \ln(x^2+36) + C$

c) [**5**] $\int_1^{\infty} \frac{\ln(x)}{x} dx$ con $u = \ln(x); du = \frac{1}{x}dx$

se tiene $\int \frac{\ln(x)}{x} dx = \int u\,du = \frac{u^2}{2} = \frac{1}{2}[\ln(x)]^2$

Comprobación: $\frac{d}{dx}\left(\frac{1}{2}[\ln(x)]^2\right) = \frac{1}{x}\ln x$

$\int_1^{\infty} \frac{\ln(x)}{x} dx = \lim_{b\to\infty} \frac{1}{2}[\ln(b)]^2 - \frac{1}{2}[\ln(1)]^2 = \infty$

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(9) [10] Calcular el valor aproximado de $\int_{0.1}^{0.3} \frac{\sin(x)}{x} dx$ utilizando un polinomio de Taylor o Maclaurin de grado 5.

RESPUESTA.

$\sin(x) = \sin(0) + \cos(0)x - \frac{\sin(0)}{2!}x^2 - \frac{\cos(0)}{3!}x^3 + \frac{\sin(0)}{4!}x^4 + \frac{\cos(0)}{5!}x^5 + R \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

$\int \frac{\sin(x)}{x} dx \approx \int \frac{1}{x}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5\right)dx = \frac{1}{600}x^5 - \frac{1}{18}x^3 + x$

$\int_{0.1}^{0.3} \frac{1}{x}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5\right)dx = \left[\frac{1}{600}x^5 - \frac{1}{18}x^3 + x\right]_{0.1}^{0.3} =$

$\left(\frac{1}{600}(0.3)^5 - \frac{1}{18}(0.3)^3 + 0.3\right) - \left(\frac{1}{600}(0.1)^5 - \frac{1}{18}(0.1)^3 + 0.1\right) = 0.19856$